ON THE CONVERGENCE OF A FAMILY OF WEIERSTRASS-TYPE ROOT-FINDING METHODS

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Abstract

In 1985, Kyurkchiev and Andreev introduced and studied a family of Weierstrass-type iterative methods with arbitrary order of convergence for computing all zeros of a polynomial simultaneously. In this paper, we present a new convergence theorem for this family, which is an improvement of Kyurkchiev and Andreev's result.

Key words: simultaneous methods, Weierstrass method, accelerated convergence, local convergence, error estimates

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1. Introduction. Let \((\mathbb{K}, | \cdot |)\) be a normed field and \(f \in \mathbb{K}[z]\) be a polynomial of degree \(n \geq 2\) with simple zeros. In this paper, we obtain a new local convergence theorem for a family of Weierstrass-type iterative methods for approximating all zeros of \(f\) simultaneously. This family was introduced by KYURKCHIEV and ANDREEV \([1]\) in 1985. The first member of their family is the famous method of WEIERSTRASS \([2]\) and the second one is a method of NOUREIN \([3]\). Our theorem generalizes, improves and complements the result of Kyurkchiev and Andreev \([1]\). In the case of the Weierstrass method the new theorem coincides with the result of PROINOV \([4]\).

2. Notations. A vector \(\xi \in \mathbb{K}^n\) is said to be a root-vector of \(f\) if and only if \(f(z) = a_0 \prod_{i=1}^n (z - \xi_i)\) for all \(z \in \mathbb{K}\), where \(a_0 \in \mathbb{K}\). We denote with \(\text{sep}(f)\) the separation number of \(f\) which is defined to be the minimum distance between two zeros of \(f\).

We endow the vector space \(\mathbb{K}^n\) with the norm \(\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}\) for some \(1 \leq p \leq \infty\). Let \((\mathbb{R}^n, \| \cdot \|_p)\) be equipped with coordinate-wise ordering \(\preceq\) defined by \(x \preceq y\) if and only if \(x_i \leq y_i\) for all \(i \in I_n\). Here and throughout, we denote by \(I_n\) the set of indices \(1, \ldots, n\). Then \((\mathbb{R}^n, \| \cdot \|_p)\) is a solid vector space. We define
a cone norm in $\mathbb{K}^n$ with values in $\mathbb{R}^n$ by $\|x\| = (|x_1|, \ldots, |x_n|)$. Then $(\mathbb{K}^n, \|\cdot\|, \preceq)$ is a cone normed space over $\mathbb{R}^n$ (see e.g. Proinov [5]).

We use the function $d: \mathbb{K}^n \rightarrow \mathbb{R}^n$ defined by $d_i(x) = \min\{x_i - x_j\}$ for every $i \in I_n$. For two vectors $x, y \in \mathbb{K}^n$ we denote by $\frac{x}{y}$ the vector in $\mathbb{R}^n$ defined by $\frac{x}{y} = (\frac{|x_1|}{y_1}, \ldots, \frac{|x_n|}{y_n})$ provided that $y$ has only nonzero components.

Given $p$ such that $1 \leq p \leq \infty$ we denote by $q$ the conjugate exponent of $p$, i.e. $q$ is defined by means of $1 \leq q \leq \infty$ and $1/p + 1/q = 1$.

Given vector $x \in \mathbb{K}^n$, $x_i$ denotes the $i$th component of $x$. In particular, if $F$ is a map with values in $\mathbb{K}^n$, then $F_i(x) = \text{the } i\text{th component of the vector } F(x)$.

3. **Weierstrass method.** In 1891, Weierstrass [2] introduced his famous iterative method for simultaneous computing of all zeros of $f$. The Weierstrass method is defined in the vector space $\mathbb{K}^n$ by the following iteration:

\begin{equation}
\label{eq:weierstrass}
x^{(k+1)} = x^{(k)} - W(x^{(k)}), \quad k = 0, 1, 2, \ldots,
\end{equation}

where the operator $W: D \subset \mathbb{K}^n \rightarrow \mathbb{K}^n$ is defined by

\begin{equation}
\label{eq:operator}
W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - x_j)} \quad (i = 1, \ldots, n),
\end{equation}

where $a_0 \in \mathbb{K}$ is the leading coefficient of $f$ and the domain $D$ of $W$ is the set of all vectors in $\mathbb{K}^n$ with distinct components.

**Definition 3.1.** Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ with $n$ simple zeros in $\mathbb{K}$, $\xi$ be a root-vector of $f$ and $1 \leq p \leq \infty$. Define the function $E: \mathbb{K}^n \rightarrow \mathbb{R}_+$ by

\begin{equation}
\label{eq:weierstrass_function}
E(x) = E_f(x) = \frac{\|x - \xi\|}{\|d(\xi)\|}_p.
\end{equation}

Recently Proinov [4] has proved the following convergence theorem for the Weierstrass method (1), which improves the results of Dochev [9], Kjurkchiev, Markov [7], Yakoubsohn [9], Proinov and Petkova [9].

**Theorem 3.1 (Proinov [4]).** Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has $n$ simple zeros in $\mathbb{K}$, $\xi$ be a root-vector of $f$ and $1 \leq p \leq \infty$. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial guess satisfying

\begin{equation}
\label{eq:initial_guess}
E(x^{(0)}) < R = \frac{2^{1/(n-1)} - 1}{2^1/q (2^{1/(n-1)} - 1) + (n-1)^{-1/p}}.
\end{equation}

Then the Weierstrass iteration (1) is well defined and converges quadratically to $\xi$ with error estimates

\begin{equation}
\label{eq:convergence}
\|x^{(k+1)} - \xi\| \leq \lambda^k \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \leq \lambda^{k-1} \|x^{(0)} - \xi\|
\end{equation}

for all $k \geq 0$, where $\lambda = \phi(E(x^{(0)}))$ and the real function $\phi$ is defined by

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\[ \phi(t) = \left(1 + \frac{t}{(n-1)^{1/p}(1-2^{1/q}t)}\right)^{n-1} - 1. \]

4. Kyurkchiev and Andreev’s family of iterative methods. In the sequel, we use the binary relation \( \# \) defined on \( \mathbb{K}^n \) by

\[ x \# y \iff x_i \neq y_j \text{ for all } i, j \in I_n \text{ with } i \neq j. \]

**Definition 4.1.** Let \( f \in \mathbb{K}[z] \) be a polynomial of degree \( n \geq 2 \). Define the sequence of functions \( T^{(N)}: D_N \subset \mathbb{K}^n \to \mathbb{K}^n \) \((N = 0, 1, 2, \ldots)\) recursively by setting \( T^{(0)}(x) \equiv x \) and

\[ T^{(N+1)}_i(x) = x_i - \frac{f(x_i)}{\prod_{j \neq i}(x_i - T^{(N)}_j(x))} \quad (i = 1, \ldots, n), \]

where the sequence of the domains \( D_N \) is also defined recursively by setting \( D_0 = \mathbb{K}^n \) and \( D_{N+1} = \{x \in D_N: x \# T^{(N)}(x)\} \).

Given \( N \in \mathbb{N} \), the \( N \)th method of Kjurkchiev and Andreev’s family can be defined by the following iteration:

\[ x^{(k+1)} = T^{(N)}(x^{(k)}), \quad k = 0, 1, 2, \ldots. \]

In the case \( N = 1 \) the method (9) coincides with the Weierstrass method (1). If \( N = 2 \), then the method (9) coincides with a method of Nourein \([3]\).

Kyurkchiev and Andreev \([1]\) proved the following convergence theorem for the family of methods (9). This theorem can also be found in \([10]\) and \([11]\).

**Theorem 4.1** (Kyurkchiev and Andreev \([1]\)). Let \( f \in \mathbb{C}[z] \) be a polynomial of degree \( n \geq 2 \) which has only simple zeros, \( \xi \in \mathbb{C}^n \) be a root-vector of \( f \) and \( N \geq 1 \). Let \( 0 < h < 1 \) and \( c > 0 \) be such that

\[ \frac{ch(1 + e^2)}{\delta - c} \leq 1 \quad \text{and} \quad 0 < \frac{nce^2}{\delta - ch(2 + e^2)} < 1, \]

where \( \delta = \text{sep}(f) \). Suppose \( x^{(0)} \in \mathbb{C}^n \) is an initial guess satisfying the condition

\[ \|x^{(0)} - \xi\|_{\infty} \leq ch. \]

Then the Weierstrass-type iteration (9) converges to \( \xi \) with error estimate

\[ \|x^{(k)} - \xi\|_{\infty} \leq ch^{(N+1)k} \quad \text{for all } k \geq 0. \]

5. Main result.

**Definition 5.1.** Let \( 1 \leq p \leq \infty \). Define the sequence of real functions \( \phi_N \)
is well defined and converges to $\|x\|$ for all $k \geq 0$. Then the Weierstrass-type iteration (9) is well defined and converges to $x$ with error estimates

$$E(x(0)) = \left\| \frac{x(0) - \xi}{d(\xi)} \right\|_p < R = \frac{2^{1/(n-1)} - 1}{2^{1/(n-1)} - 1 + (n-1)^{-1/p}}.$$ 

Then the Weierstrass-type iteration (9) is well defined and converges to $x$ with error estimates

$$\|x^{(k+1)} - \xi\| \leq \lambda^{(N+1)^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \leq \lambda^{((N+1)^k-1)/N} \|x^{(0)} - \xi\|$$

for all $k \geq 0$, where $\lambda = \phi_N(E(x^{(0)}))$ and $\phi_N$ is defined by Definition 5.1.

**Corollary 5.1.** Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ with $n$ simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root-vector of $f$, $N \geq 1$ and $1 \leq p \leq \infty$. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial guess satisfying (14). Then the Weierstrass-type iteration (9) is well defined and converges to $x$ with error estimates

$$\|x^{(k+1)} - \xi\| \leq \lambda^{N(N+1)^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \leq \lambda^{(N+1)^k-1} \|x^{(0)} - \xi\|$$

for all $k \geq 0$, where $\lambda = \phi(E(x^{(0)}))$ and the real function $\phi$ is defined by (6).

**Proof.** It follows from Theorem 5.1 and the inequality $\phi_N(t) \leq \phi(t)^N$ which holds for all $t \in [0, R]$ (see Lemma 6.1).

**Corollary 5.2.** Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ with $n$ simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root-vector of $f$, $N \geq 1$, $1 \leq p \leq \infty$, and let

$$0 < h < 1 \quad \text{and} \quad 0 < c \leq R \text{sep}(f),$$

where $R$ is defined in (14). Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial guess satisfying

$$\|x^{(0)} - \xi\|_p \leq ch.$$ 

Then the Weierstrass-type iteration (9) is well defined and converges to $x$ with error estimates

$$\|x^{(k+1)} - \xi\| \leq h^{N(N+1)^k} \|x^{(k)} - \xi\|, \quad \|x^{(k)} - \xi\| \leq h^{(N+1)^k-1} \|x^{(0)} - \xi\|,$$

$$\|x^{(k)} - \xi\|_p \leq ch^{(N+1)^k} \quad \text{for all} \quad k \geq 0.$$
Proof. From (17) and (18), we obtain $E(x^{(0)}) \leq hR < R$. Then, according to Corollary 5.1, the Weierstrass-type iteration (9) converges to $\xi$ with error estimates (16), which imply the error estimates (19). Indeed, we have

$$\lambda = \phi(E(x^{(0)})) \leq \phi(hR) \leq h\phi(R) = h$$

because $0 < h < 1$, $\phi(R) = 1$ and $\phi_N$ is a quasi-homogeneous function of the first degree on $[0, R]$ (see Lemma 6.1). Now the estimate (20) follows from the second estimate in (19) and (18).

Remark 5.1. Corollary 5.2 improves and complements Theorem 4.1. Indeed, it is easy to check that (10) is a stronger assumption than (17) in the case $p = \infty$.

6. Proof of the main result. Throughout this section, we follow the terminology of [4].

Definition 6.1. Given $N \geq 0$ and $1 \leq p \leq \infty$, define the function $\varphi_N$ by

$$\varphi_N(t) = t\phi_N(t).$$

Lemma 6.1. Let $N \geq 0$ and $1 \leq p \leq \infty$. Then:

(i) $\phi_N$ is a quasi-homogeneous of degree $N$ on $[0, 1/2^{1/q})$ and $\phi_N(R) = 1$;

(ii) $\phi_{N+1}(t) \leq \phi(t)\phi_N(t)$ and $\phi_N(t) \leq \phi(t)^{N}$ for all $t \in [0, R]$;

(iii) $\varphi_N$ is a gauge function of order $N + 1$ on $[0, R]$.

Proof. Claim (ii) follows from (i) and (6). Claim (iii) follows from (i) and Proposition 2.4 of [4]. We shall prove (i) by induction on $N$. The case $N = 0$ is obvious. Assume that for some $N \geq 0$ the function $\phi_N$ is well defined on $\Delta = [0, 1/2^{1/q})$ and satisfies (i). It follows from Definition 5.1 and $R < 1/2^{1/q}$, that $\phi_{N+1}$ is well defined on $\Delta$. From Example 2.2 of [4], we get that $\phi_{N+1}$ is quasi-homogeneous of degree $N + 1$ on $\Delta$. Finally, from the definition of $\phi_{N+1}$ and $\phi_N(R) = 1$, we obtain

$$\phi_{N+1}(R) = \phi(R) = 1$$

which completes the proof.

Lemma 6.2. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root-vector of $f$ and $N \geq 0$. If $x \in D_{N+1}$, then for every $i \in I_n$,

$$T_i^{(N+1)}(x) - \xi_i = \left(1 - \prod_{j \neq i} (1 + u_j)\right)(x_i - \xi_i),$$

where $u_j \in \mathbb{K}$ is defined by

$$u_j = \frac{T_j^{(N)}(x) - \xi_j}{x_i - T_j^{(N)}(x)}.$$
Proof. From Definition 4.1, we get

$$T_i^{(N+1)}(x) - \xi_i = x_i - \xi_i - \frac{f(x_i)}{\prod_{j \neq i}(x_i - T_j^{(N)}(x))}$$

$$= x_i - \xi_i - (x_i - \xi_i) \prod_{j \neq i} \frac{x_i - \xi_j}{x_i - T_j^{(N)}(x)} = \left(1 - \prod_{j \neq i}(1 + u_j)\right) (x_i - \xi_i)$$

which completes the proof.

Lemma 6.3. Let $u, v, \xi \in \mathbb{K}^n$ and $1 \leq p \leq \infty$. If $\xi$ is a vector with distinct components such that

$$\|v - \xi\| \leq \|u - \xi\|,$$

then for all $i, j \in I_n$,

$$|u_i - v_j| \geq \left(1 - 2^{1/\|\xi\|p}\right) |\xi_i - \xi_j|.$$  \hspace{1cm} (25)

Proof. We consider the nontrivial case $i \neq j$. From the triangle inequality in $\mathbb{K}$, (24) and Hölder’s inequality, we obtain

$$|u_i - v_j| \geq |\xi_i - \xi_j| - |u_i - \xi_i| - |v_j - \xi_j| \geq |\xi_i - \xi_j| - |u_i - \xi_i| - |u_j - \xi_j|$$

$$\geq \left(1 - \frac{|u_i - \xi_i|}{d_i(\xi)} - \frac{|u_j - \xi_j|}{d_j(\xi)}\right) |\xi_i - \xi_j| \geq \left(1 - 2^{1/\|\xi\|p}\right) |\xi_i - \xi_j|$$

which completes the proof.

Lemma 6.4. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has $n$ simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root-vector of $f$, $N \geq 0$ and $1 \leq p \leq \infty$. Suppose $x \in \mathbb{K}^n$ is a vector satisfying the following condition

$$E(x) < R,$$

where the number $R$ is defined in (4). Then $x \in D_N$ and

$$\|T_j^{(N)}(x) - \xi\| \leq \phi_N(E(x))\|x - \xi\|.$$  \hspace{1cm} (27)

Proof. We shall prove statements by induction on $N$. The case $N = 0$ is obvious. Assume that $x \in D_N$ and (27) hold for some $N \geq 0$.

Let $u = x$ and $v = T_j^{(N)}(x)$. It follows from Lemma 6.1(i) that $\phi_N(t) \leq 1$ for all $t \in [0, R]$. Then it follows from (27) that $\|v - \xi\| \leq \|u - \xi\|$. By Lemma 6.3 with $u = x$ and $v = T_j^{(N)}(x)$ and taking into account that $R < 1/2^{1/\|\xi\|p}$, we obtain

$$|x_i - T_j^{(N)}(x)| \geq (1 - 2^{1/\|\xi\|p}E(x)) d_j(\xi) > 0$$  \hspace{1cm} (28)

for every $j \neq i$. Consequently, $x \neq T_j^{(N)}(x)$ which proves that $x \in D_{N+1}$.
To show that (27) holds for $N + 1$, we have to prove that

$$|T^{(N+1)}_i(x) - \xi_i| \leq \phi_{N+1}(E(x)) |x_i - \xi_i| \quad \text{for all } i \in I_n.$$  

Let $i \in I_n$ be fixed. We consider the vector $u = (u_j)_{j \neq i} \in \mathbb{K}^{n-1}$, where $u_j$ is defined by (23). It follows from (27) and (28) that

$$|u_j| = \frac{|T_j^{(N)}(x) - \xi_j|}{|x_i - T_j^{(N)}(x)|} \leq \frac{\phi_N(E(x))}{1 - 2^{1/q}E(x)} \frac{|x_j - \xi_j|}{d_j(\xi)}.$$ 

Taking the $p$-norm, we obtain

$$\|u\|_p \leq \frac{E(x) \phi_N(E(x))}{1 - 2^{1/q}E(x)}.$$ 

Then, from Lemma 6.2, Proposition 5.5 of [4], (30) and Definition 5.1, we obtain

$$|T^{(N+1)}_i(x) - \xi_i| \leq \left[1 + \frac{\|u\|_p}{(n-1)^{1/p}}\right]^{n-1} |x_i - \xi_i|$$

$$\leq \left[1 + \frac{E(x) \phi_N(E(x))}{(n-1)^{1/p}(1 - 2^{1/q}E(x))}\right]^{n-1} |x_i - \xi_i| = \phi_{N+1}(E(x)) |x_i - \xi_i|$$

which proves (29). This completes the proof.

**Lemma 6.5.** Suppose $N \geq 1$. Let $T^{(N)}: D_N \subset \mathbb{K}^n \to \mathbb{K}^n$ and $E: D_N \to \mathbb{R}_+$ be defined by Definitions 4.1 and 3.1, respectively. Then:

(i) $E: D_N \to \mathbb{R}_+$ is a function of initial conditions of $T^{(N)}$ with a strict gauge function $\varphi_N$ of order $N + 1$ on $J = [0, R)$.

(ii) $T^{(N)}: D_N \to \mathbb{K}^n$ is an iterated contraction at $\xi$ with respect to $E$ with control function $\phi_N$.

(iii) Every point $x^{(0)} \in \mathbb{K}^n$ such that $E(x^{(0)}) \in J$ is an initial point of $T^{(N)}$.

**Proof.** From Lemma 6.4, we get $E(T^{(N)}(x)) \leq \varphi_N(E(x))$ for all $x \in \mathbb{K}^n$ such that $E(x) \in J$. From this and Lemma 6.1(iii), we conclude that (i) holds. Claim (ii) follows from Lemma 6.4. Claim (iii) follows from Lemma 6.4 and Proposition 2.7 of [4].

**Proof of Theorem 1.1.** It follows from Lemma 6.5 and Corollary 3.4 of [4].

**REFERENCES**


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