

A MORE FLEXIBLE COUNTERPART OF A  
HUANG–KOTZ'S COPULA-TYPE

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Abstract

We propose a more flexible symmetric counterpart of the Huang–Kotz's copula of the 1st type. Both the counterpart and Huang–Kotz's copula of the 1st type provide the same improvement in the correlation level. Moreover, the proposed copula includes special cases of many other extensions of the Farlie–Gumbel–Morgenstern (FGM) copula.

**Key words:** FGM copula, Huang–Kotz FGM copula, iterated FGM copula, Spearman's Rho, Kendall's Tau

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**1. Introduction.** A bivariate copula is a bivariate distribution function defined on the unit square  $I^2 = [0, 1]^2$ , which has uniform marginals on  $[0, 1]$ . Copula is mainly used to describe the dependence between random variables (RVs). Any bivariate function,  $C(u, v)$ , is a copula if and only if (cf. [1])

$$(1.1) \quad C(u, 0) = C(0, v) = 0, C(u, 1) = u, C(u, v) \leq 1, C(1, v) = v, \forall (u, v) \in I^2,$$

$$(1.2) \quad \text{and } C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0,$$

$$\forall (u_i, v_i) \in I^2, i = 1, 2, u_1 \leq u_2, v_1 \leq v_2.$$

Moreover, if  $C(u, v)$  is twice differentiable, then we should have

$$(1.3) \quad \frac{\partial^2 C(u, v)}{\partial u \partial v} = c(u, v) \geq 0,$$

where  $c(u, v)$  is called copula density. For the interrelation between the 2-increasing property (1.2) and positive copula density (1.3), see [2].

The FGM copula is defined by

$$(1.4) \quad C_{FGM}(u, v; a) = uv\{1 + a(1 - u)(1 - v)\}, \quad a \in [-1, 1], \quad (u, v) \in I^2.$$

While the copula (1.4) is a flexible family and valuable in many applications, a well-known limitation of this copula is the low dependence level it permits between RVs, Spearman's Rho  $\rho \in [-1/3, 1/3]$  and Kendall's Tau  $\tau \in [-2/9, 2/9]$ . Several extensions to the FGM copula have been introduced in the literature, where most of these extensions aimed to improve the correlation level. Some examples of these important extensions are [3-6]. Recently, most of these extensions were studied in different important aspects by ABD ELGAWAD et al. [7-9] and ALAWADY et al. [10,11].

HUANG and KOTZ [12] proposed two analogous extensions, the 1st HK-FGM and 2nd HK-FGM types which are defined, respectively, by

$$(1.5) \quad C_{HK-FGM1}(u, v; a, b) = uv \left[ 1 + a(1 - u^b)(1 - v^b) \right], \quad b > 0, \\ a \in \left[ -\frac{1}{b^2}, \frac{1}{b} \right],$$

$$(1.6) \quad C_{HK-FGM2}(u, v; a, b) = uv \left[ 1 + a(1 - u)^b(1 - v)^b \right], \quad b > 1, \\ a \in \left[ -1, \left( \frac{b+1}{b-1} \right)^{b-1} \right],$$

with  $\rho \in [-1/3, 3/8]$  and  $\rho \in [-1/3, 391/1000]$ , respectively. The difference between the maximal positive correlations of the two types (1.5) and (1.6) is marginal. For more details about the Huang-Kotz copulas, see the recent works [13,14].

EBAID et al. [15] proposed the following new extended FGM copula

$$(1.7) \quad C(u, v) = uv [1 + a(1 - u)(1 - v)(1 - bu)(1 - bv)], \quad b \in [0, 1).$$

Ebaid et al. [15] claimed that, the admissible range of the parameters is

$$(1.8) \quad a \in \left[ -1, \frac{1}{1-b} \right] \quad \text{and} \quad b \in [0, 1).$$

Moreover,  $\rho \in [-1/3, 1]$  and  $\tau \in [-2/9, 1]$ . BARAKAT et al. [16] proved that the admissible range (1.8) and the claim about the correlations are wrong. Moreover,

Barakat et al. [16] showed that the copula (1.7), whenever  $b \in [0, 1)$ , does not increase the maximum positive correlations for the FGM copula (1.4).

In this paper, we modify the copula (1.7) by elongating the range of the parameter  $b$  to  $[-\infty, \infty]$ , where  $\pm\infty$  is interpreted as  $\lim_{b \rightarrow \pm\infty}$ . The modified copula would be considered a counterpart of the Huang and Kotz's copula of the 1st type defined by (1.5), in the sense that both have two shape parameters and provide the same improvement of the positive correlation between the dependent variables compared with the FGM copula. However, the modified copula has an evident preference over the copula (1.5) because the new copula has a simpler functional form more than the copula (1.5), for being that in order to get it, as an extension of the classical FGM copula (1.4), we used an extra shape parameter as a multiplicative factor, while to get the copula (1.5), the extra shape parameter is used as an exponent. Besides this evident motivation, the modified copula includes some special cases of other extensions of the FGM copula.

Moreover, Ebaid et al. [15] used the copula (1.7), with a wrong admissible range (cf. [16]), to estimate the reliability in a dependent stress-strength model with an application to the Egyptian finance system. There is no doubt that this important application cannot be benefitted from as long as the admissible range of the copula (1.7) has not been determined or was specified in a wrong way. Therefore, the result of this paper enables us to use this application.

**2. The main result.** The suggested modified copula of (1.7) is defined by

$$(2.1) \quad C(u, v; a, b) = uv \{1 + a(1 - u)(1 - v)(1 + bu)(1 + bv)\}, \\ (u, v) \in I^2, b \in [-\infty, \infty].$$

Clearly, the Spearman's Rho and Kendall's Tau of the copula (2.1) can be determined by reversing the sign of  $b$  in (17) and (18) in Ebaid et al. [15] as  $\rho = \frac{a(2+b)^2}{12}$  and  $\tau = \frac{a(2+b)^2}{18}$ , respectively.

In the following theorem, we determine a subset  $\Omega$  of the admissible range of the copula (2.1), on which the copula provides an improvement of the correlation coefficients  $\rho$  and  $\tau$ .

**Theorem 2.1.** *The set  $\Omega$  is given by  $\Omega = \Omega^+ \cup \Omega^-$ , where*

$$(2.2) \quad \Omega^+ = \left\{ (a, b) : 0 \leq b \leq 1, -\frac{1}{(1+b)^2} \leq a \leq \frac{1}{(1+b)}; \right. \\ \left. \text{or } b > 1, -\frac{1}{(1+b)^2} \leq a \leq \frac{1}{(1+b)^2} \right\}, \\ \Omega^- = \left\{ (a, b) : -2 \leq b \leq 0, -1 \leq a \leq 0; \right. \\ \left. \text{or } b < -2, -\frac{1}{(1+b)^2} \leq a \leq \frac{1}{(1+b)^2} \right\}.$$

**Proof.** The admissible range  $\Omega$  will be determined relying on the conditions (1.1) and (1.3). The corresponding copula density of (2.1) can be written in the form

$$(2.3) \quad c(u, v; a, b) = 1 + af(u, b)f(v, b),$$

where  $f(u, b) = 3bu^2 + 2u(1 - b) - 1$ . Clearly, the condition (1.1) is satisfied, when  $0 < b < 1$ . Furthermore, when  $0 < b < 1$ , in view of (2.3), we get  $f(0, b) = -1$  and  $f(1, b) = 3b + 2(1 - b) - 1 = b + 1 > 0$ . On the other hand,  $f'(u, b) = \frac{df(u, b)}{du} = 6bu + 2(1 - b) > 0$ . Thus,  $f(u, b)$  is strictly increasing. Moreover,  $\min_{0 \leq u \leq 1} f(u, b) = -1$  and  $\max_{0 \leq u \leq 1} f(u, b) = 1 + b > 0$ . Therefore, in order that, the condition (1.3) is satisfied, i.e.  $c(u, v; a, b) \geq 0$ , we must have

$$1 + a \min_{0 \leq u \leq 1} f(u, b) \times \min_{0 \leq v \leq 1} f(v, b) \geq 0 \implies 1 + a \geq 0 \implies a \geq -1,$$

$$1 + a \min_{0 \leq u \leq 1} f(u, b) \times \max_{0 \leq v \leq 1} f(v, b) \geq 0 \implies 1 - (1 + b)a \geq 0 \implies a \leq \frac{1}{1 + b},$$

$$1 + a \max_{0 \leq u \leq 1} f(u, b) \times \max_{0 \leq v \leq 1} f(v, b) \geq 0 \implies 1 + (1 + b)^2 a \geq 0 \implies a \geq \frac{-1}{(1 + b)^2}.$$

The above restrictions on  $a$  imply that  $a \in \left[ \frac{-1}{(1 + b)^2}, \frac{1}{1 + b} \right]$ . Bearing in mind that,

$$C(u, v; a, 0) = C_{FGM}(u, v; a), -1 \leq a \leq 1$$

and

$$C(u, v; a, 1) = C_{HK-FGM1}(u, v; a, 2), -\frac{1}{4} \leq a \leq \frac{1}{2},$$

we get  $(a, b) \in \Omega^+$ , for  $0 \leq b \leq 1$ , and  $a \in \left[ \frac{-1}{(1 + b)^2}, \frac{1}{1 + b} \right]$ . Now, consider the case  $b > 1$ . Clearly, while for this case the condition (1.3) is still satisfied, but the condition (1.1) is not, especially for large values of  $b$  (namely, if  $a \leq \frac{1}{1 + b}$ , we get  $\lim_{b \rightarrow \infty} C(u, v; a, b) = \infty$ ) unless we alter the upper bound of  $a$  to be  $\frac{1}{(1 + b)^2}$ , note that in this case  $\lim_{b \rightarrow \infty} C(u, v; \frac{\alpha}{(1 + b)^2}, b) = uv[1 + \alpha uv(1 - u)(1 - v)] := C_{IFGM}(u, v; 0, \alpha)$ , where  $-1 \leq \alpha \leq 3$ , and  $C_{IFGM}(u, v; a, b)$  is a single iterated FGM copula, see Proposition 2.2. Thus, we proved that  $\Omega^+ \subset \Omega$ . We turn now to the case  $b < 0$ . When  $-1 \leq b \leq 0$ , which implies  $1 \geq 1 + b \geq 0$ , and by proceeding as we have done in the case  $b > 0$ , we can check that the conditions  $a \geq -1$ ,  $a \leq \frac{1}{1 + b}$ , and  $a \geq -\frac{1}{(1 + b)^2}$  guarantee the validation of the condition (1.3) if

$a \in \left[-1, \frac{1}{1+b}\right]$ , but the condition  $a \leq \frac{1}{1+b}$  makes  $C(u, v; a, b) > 1$ , for some values of  $b$  (when the value of  $b$  is equal to  $-1$ , or close to it). In order to fix the upper bound of  $a$ , we invoke the general relation  $C(u, v; a, b) \leq 1 + a \leq 1$ , which enables us to alter the upper bound  $\frac{1}{1+b}$  by 0. On the other hand, if  $-2 \leq b < -1$ , which implies  $-1 < 1+b < 0$ , and by proceeding as we have done in the case  $b > 0$ , we can check that the condition (1.3) is satisfied if  $a \geq -1$ ,  $a \geq \frac{1}{1+b}$ , and  $a \geq -\frac{1}{(1+b)^2}$ . These conditions yield  $a \geq -\frac{1}{(1+b)^2}$ , while the upper bound of  $a$  is zero (by using the inequality  $C(u, v; a, b) \leq 1 + a \leq 1$ ). Finally, when  $b < -2$ , we can show that the condition (1.3) is satisfied if  $a \in \left[\frac{-1}{(1+b)^2}, \frac{1}{1+b}\right]$ . On the other hand, in order that the condition (1.1) is satisfied (especially for large  $|b|$ ), we can alter the admissible range of  $a$  to be  $\left[\frac{-1}{(1+b)^2}, \frac{1}{(1+b)^2}\right]$ . This proves that  $\Omega^- \subset \Omega$ , as required to complete the proof.  $\square$

**Remark 2.1.** We can easily check that the improvement of the correlation attains only on  $\Omega^+$ . Namely, at  $b = 1, a = \frac{1}{2}$ , we get  $\rho = 3/8$ . On the other hand, no improvement can be gained on  $\Omega^-$ , which coincides with the remark of Barakat et al. [16] that the copula (1.7) does not improve the correlation.

**Proposition 2.2.** *It is easy to check the validity of the following relations and properties:*

- (1)  $C(u, v; a, 0) = C_{FGM}(u, v; a), -1 \leq a \leq 1,$
- (2)  $C(u, v; a, 1) = C_{HK-FGM1}(u, v; a, 2), -\frac{1}{4} \leq a \leq \frac{1}{2},$
- (3)  $C(u, v; a, -1) = C_{HK-FGM2}(u, v; a, 2), -1 \leq a \leq 3,$
- (4)  $\lim_{b \rightarrow \pm\infty} C(u, v; \frac{\alpha}{(1+b)^2}, b) = C_{IFGM}(u, v; 0, \alpha),$  where  $-1 \leq \alpha \leq 3$  and

$$\begin{aligned}
 C_{IFGM}(u, v; a, b) &= uv \{1 + a(1-u)(1-v) + buv(1-u)(1-v)\}, \\
 &- 1 \leq a \leq 1, a + b \geq -1, b \leq \frac{3 - a + \sqrt{9 - 6a - 3a^2}}{2},
 \end{aligned}$$

*is the single iterated FGM copula, which was introduced by Huang and Kotz [17] and further studied by ALAWADY et al. [18], BARAKAT and HUSSEINY [19], and BARAKAT et al. [20].*

**3. Conclusion.** In this paper, we revisited an extended Farlie–Gumbel–Morgenstern (FGM) copula, which was previously studied and published by Ebaid et al. [15], and then rectified by Barakat et al. [16]. The admissible range derived by Ebaid et al. [15] and the claim about the correlation were demonstrated to be incorrect in [16]. The corrected admissible range of this copula was obtained in this paper, and it was shown to be a more flexible counterpart of a Huang–Kotz’s

copula of the first type defined by (1.5). Furthermore, it was shown that this copula produces the same improvement in correlation as Huang–Kotz’s copula of the 1st type. Finally, we demonstrated that this copula includes many other FGM copula extensions as special cases.

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