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THIRD NATURAL CONNECTION ON RIEMANNIAN Π-MANIFOLDS

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Abstract

A natural connection whose torsion tensor is a 3-form is defined and called the third natural connection on a Riemannian Π-manifold. In the cases of existence of this connection, a relation between it and the known first and second natural connections on the considered manifolds is established. An explicit example of dimension 5 is given in confirmation of the proven assertions.

Key words: third natural connection, second natural connection, first natural connection, affine connection, natural connection, Riemannian Π-manifolds

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1. Introduction. We study the differential geometry of Riemannian Π-manifolds, also known as almost paracontact almost paracomplex Riemannian manifolds [1–3]. These odd-dimensional manifolds have a traceless induced almost product structure on the paracontact distribution and the restriction on the paracontact distribution of the almost paracontact structure is an almost paracomplex structure.

In the present paper we continue the study of natural connections on the considered manifolds as an important part of the geometry of manifolds with additional tensor structures (e.g. [4–9]). Following the introduction and investigation of the first natural connection $\dot{D}$ in [10] and the second natural connection $\ddot{D}$ in

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[11], here we introduce a natural connection $\dddot{D}$ on a Riemannian Π-manifold whose torsion tensor is a 3-form and call it the third natural connection.

The paper is structured as follows. In Section 1 and Section 2, we give an introduction and preliminary background facts about Riemannian Π-manifolds, respectively. In Section 3, we recall some definitions and assertions for the first and second natural connection necessary for further investigations. Section 4 is devoted to the definition of $\dddot{D}$. Moreover, we obtain a relation between the Nijenhuis tensor N and the torsion tensor of $\dddot{D}$ and we prove a necessary and sufficient condition for existence of $\dddot{D}$. Finally, we establish that for the manifolds where $\dddot{D}$ exists, $\dot{D}$ is the average connection of $\ddot{D}$ and $\dddot{D}$. In the final Section 5, we present an explicit 5-dimensional example in confirmation of the proven theory.

2. Riemannian Π-Manifolds. Let $(\mathcal{M}, \phi, \xi, \eta, g)$ be a Riemannian Π-manifold, i.e. $\mathcal{M}$ stands for a $(2n+1)$-dimensional differentiable manifold equipped with a Riemannian metric g and a Riemannian Π-structure (ϕ, ξ, η), where ϕ is a (1,1)-tensor field, ξ is a Reeb vector field and η is its dual 1-form, such that the following basic identities

$$\begin{gathered} \phi\xi = 0, \qquad \phi^2 = I - \eta \otimes \xi, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1, \\ \operatorname{tr}\phi = 0, \qquad g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y), \end{gathered} \tag{2.1}$$

and their immediately derived properties

$$g(\phi x, y) = g(x, \phi y), \quad g(x, \xi) = \eta(x), \quad g(\xi, \xi) = 1, \quad \eta(\nabla_x \xi) = 0, \tag{2.2}$$

are valid, where I and ∇ denote the identity and the Levi-Civita connection of g, respectively [2,12]. In the latter equalities and further, x, y, z stand for arbitrary differentiable vector fields on $\mathcal{M}$ or tangent vectors at a point of $\mathcal{M}$.

The $(0,3)$-tensor field F, defined by

$$F(x, y, z) = g\big((\nabla_x \phi)\, y, z\big), \tag{2.3}$$

is fundamentally important in the geometry of the considered manifolds. Moreover, the following general properties of F are valid: [1]

$$\begin{aligned} &F(x, y, z) = F(x, z, y) = -F(x, \phi y, \phi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi), \\ &F(x, y, \phi z) = -F(x, \phi y, z) + \eta(z)F(x, \phi y, \xi) + \eta(y)F(x, \phi z, \xi). \end{aligned} \tag{2.4}$$

Lemma 2.1 ([2])**.** *The following identities are valid:*

1) $(\nabla_x \eta)(y) = g(\nabla_x \xi, y)$, 2) $\eta(\nabla_x \xi) = 0$, 3) $F(x, \phi y, \xi) = -(\nabla_x \eta)(y)$.

A classification of Riemannian Π-manifolds with respect to F, consisting of eleven basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, …, $\mathcal{F}_{11}$, is given in [1]. The intersection of the basic classes is the special class $\mathcal{F}_0$ determined by $F = 0$. We focus on two of these classes that are relevant for further investigations [1,2]:

$$
\begin{aligned}
\mathcal{F}_3 : \quad & F(\xi,y,z)=0,\ F(x,\xi,z)=0, \\
& F(x,y,z)+F(y,z,x)+F(z,x,y)=0; \\
\mathcal{F}_7 : \quad & F(x,y,z)=F(x,y,\xi)\eta(z)+F(x,z,\xi)\eta(y), \\
& F(x,y,\xi)=-F(y,x,\xi)=F(\phi x,\phi y,\xi).
\end{aligned} \tag{2.5}
$$

In [2], the Nijenhuis tensor N and the associated Nijenhuis tensor $\widehat{N}$ of type $(1,2)$ for the Π-structure on $\mathcal{M}$ are determined by

$$
\begin{aligned}
N(x,y) = &\left(\nabla_{\phi x}\phi\right)y-\phi\left(\nabla_x\phi\right)y-\left(\nabla_x\eta\right)(y)\xi \\
&-\left(\nabla_{\phi y}\phi\right)x+\phi\left(\nabla_y\phi\right)x+\left(\nabla_y\eta\right)(x)\xi,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{N}(x,y) = &\left(\nabla_{\phi x}\phi\right)y-\phi\left(\nabla_x\phi\right)y-\left(\nabla_x\eta\right)(y)\xi \\
&+\left(\nabla_{\phi y}\phi\right)x-\phi\left(\nabla_y\phi\right)x-\left(\nabla_y\eta\right)(x)\xi.
\end{aligned}
$$

The corresponding $(0,3)$-tensors of N and $\widehat{N}$ on $(\mathcal{M},\phi,\xi,\eta,g)$ are defined by

$$
N(x,y,z)=g\left(N(x,y),z\right), \qquad \widehat{N}(x,y,z)=g\left(\widehat{N}(x,y),z\right) \tag{2.6}
$$

and expressed by means of F through the equalities [2]:

$$
\begin{aligned}
N(x,y,z) &= F(\phi x,y,z)-F(\phi y,x,z)-F(x,y,\phi z)+F(y,x,\phi z) \\
&\quad +\eta(z)\left\{F(x,\phi y,\xi)-F(y,\phi x,\xi)\right\}, \\
\widehat{N}(x,y,z) &= F(\phi x,y,z)+F(\phi y,x,z)-F(x,y,\phi z)-F(y,x,\phi z) \\
&\quad +\eta(z)\left\{F(x,\phi y,\xi)+F(y,\phi x,\xi)\right\}.
\end{aligned} \tag{2.7}
$$

Let us recall that if the Lie derivative of g with respect to ξ vanishes, i.e. $(\mathfrak{L}_\xi g)(x,y)=0$, then ξ is called a Killing vector field regarding g.

Lemma 2.2. *The class $\widehat{\mathcal{U}}_0=\mathcal{F}_3\oplus\mathcal{F}_7$ with the property $\widehat{N}=0$ is the class of the Riemannian Π-manifolds $(\mathcal{M},\phi,\xi,\eta,g)$ with a Killing vector field ξ and a vanishing cyclic sum of the fundamental tensor F.*

Proof. Using (2.5) and the fact that $\mathfrak{L}_\xi g=0$, we prove the lemma. □

3. First and second natural connections on Riemannian Π-manifolds. In [10], we defined a non-symmetric natural connection and called it the first natural connection on a Riemannian Π-manifold $(\mathcal{M},\phi,\xi,\eta,g)$. After that, in [11], we introduced a natural connection, determined by a property of its torsion tensor, and called it the second natural connection on $(\mathcal{M},\phi,\xi,\eta,g)$. We recall some definitions and assertions necessary for further investigations.

3.1. First natural connection on Riemannian* Π*-manifolds.

Definition 3.1 ([10]). An affine connection D on a Riemannian Π-manifold $(\mathcal{M},\phi,\xi,\eta,g)$ is called a natural connection for the Riemannian Π-structure (ϕ,ξ,η,g) if this structure is parallel with respect to D, i.e. $D\phi = D\xi = D\eta = Dg = 0$.

Let us denote by T the torsion tensor of an arbitrary affine connection D, i.e.

$$T(x,y) = D_x y - D_y x - [x,y], \qquad T(x,y,z) = g(T(x,y),z). \tag{3.1}$$

By t, t^* and $\hat{t}$, we denote the associated 1-forms of T with respect to a basis $\{\xi;e_i\}$ of $T_p\mathcal{M}$ $(i=1,2,\dots,2n; p\in\mathcal{M})$:

$$t(x) = g^{ij}T(x,e_i,e_j), \qquad t^*(x) = g^{ij}T(x,e_i,\phi e_j), \qquad \hat{t}(x) = T(x,\xi,\xi). \tag{3.2}$$

Let Q stand for the potential of D with respect to ∇:

$$D_x y = \nabla_x y + Q(x,y), \qquad Q(x,y,z) = g\left(Q(x,y),z\right). \tag{3.3}$$

Proposition 3.2 ([10]). *An affine connection D is a natural connection on a Riemannian* Π*-manifold if and only if the following properties hold:*

$$Q(x,y,\phi z) - Q(x,\phi y,z) = F(x,y,z), \qquad Q(x,y,z) = -Q(x,z,y).$$

Definition 3.3 ([10]). A natural connection $\dot{D}$, defined by

$$\dot{D}_x y = \nabla_x y - \frac{1}{2}\big\{(\nabla_x\phi)\,\phi y - (\nabla_x\eta)\,y\cdot\xi\big\} - \eta(y)\nabla_x\xi,$$

is called the first natural connection on $(\mathcal{M},\phi,\xi,\eta,g)$.

Let us denote by $\dot{T}$ the torsion tensor of $\dot{D}$, i.e. analogous definitions of (3.1) are valid for $\dot{T}$.

3.2. Second natural connection on Riemannian* Π*-manifolds. Let $\ddot{T}$ stand for the torsion tensor of a natural connection $\ddot{D}$. Similarly to $\dot{T}$, the tensor $\ddot{T}$ is determined by analogous definitions of (3.1).

Definition 3.4 ([11]). A natural connection $\ddot{D}$ satisfying

$$\begin{aligned}
&\ddot{T}(x,y,z) + \ddot{T}(y,z,x) + \ddot{T}(\phi x,y,\phi z) + \ddot{T}(y,\phi z,\phi x)\\
&- \eta(x)\left\{\ddot{T}(\xi,y,z) + \ddot{T}(y,z,\xi) - \eta(y)\ddot{T}(\xi,z,\xi)\right\}\\
&- \eta(y)\left\{\ddot{T}(x,\xi,z) + \ddot{T}(\xi,z,x) + \ddot{T}(\phi x,\xi,\phi z) + \ddot{T}(\xi,\phi z,\phi x)\right\}\\
&- \eta(z)\left\{\ddot{T}(x,y,\xi) + \ddot{T}(y,\xi,x) - \eta(y)\ddot{T}(x,\xi,\xi)\right\} = 0
\end{aligned}$$

is called the second natural connection on $(\mathcal{M},\phi,\xi,\eta,g)$.

Theorem 3.5 ([11]). *The first natural connection $\dot{D}$ coincides with the second natural connection $\ddot{D}$ if and only if*

$$(\mathcal{M},\phi,\xi,\eta,g)\in\mathcal{U}_1=\mathcal{F}_1\oplus\mathcal{F}_2\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{10}\oplus\mathcal{F}_{11}.$$

Therefore, $\dot{T}$ and $\ddot{T}$ differ from each other when $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_3$ or $\mathcal{F}_7$ or to their direct sums with other classes. The forms of $\dot{T}$ and $\ddot{T}$ in these classes are [11]:

$$\begin{aligned}
&\widehat{\mathcal{U}}_0:\ \dot{T}=-\frac{1}{8}\big\{\mathfrak{S}\,N^{\mathrm{h}}+N^{\mathrm{h}}\big\}+\frac{1}{2}\big\{\eta\wedge\mathrm{d}\eta+\mathrm{d}\eta\otimes\eta\big\},\\
&\qquad\ \ \ddot{T}=-\frac{1}{4}N^{\mathrm{h}}+\mathrm{d}\eta\otimes\eta,\\
&\mathcal{F}_3:\ \dot{T}=-\frac{1}{8}\big\{\mathfrak{S}\,N^{\mathrm{h}}+N^{\mathrm{h}}\big\},\qquad \ddot{T}=-\frac{1}{4}N^{\mathrm{h}},\\
&\mathcal{F}_7:\ \dot{T}=\frac{1}{2}\big\{\eta\wedge\mathrm{d}\eta+\mathrm{d}\eta\otimes\eta\big\},\qquad \ddot{T}=\mathrm{d}\eta\otimes\eta,
\end{aligned}\tag{3.4}$$

where $N^{\mathrm{h}}(x,y,z)=N(\phi^2x,\phi^2y,\phi^2z)$ and $\mathfrak{S}$ stands for the cyclic sum by the three arguments.

4. Third natural connection on Riemannian Π-manifolds. Let $\dddot{T}$ stand for the torsion tensor of a natural connection $\dddot{D}$. Similarly to $\dot{T}$ and $\ddot{T}$, the tensor $\dddot{T}$ is determined by analogous definitions of (3.1). Let $\dddot{Q}$ denote the potential of $\dddot{D}$ with respect to ∇ and let it be defined analogously to (3.3).

Definition 4.1. A natural connection $\dddot{D}$ is called the third natural connection on $(\mathcal{M},\phi,\xi,\eta,g)$ if its torsion tensor $\dddot{T}$ is totally skew-symmetric, i.e. a 3-form.

Let us remark that the restrictions of $\dot{D}$, $\ddot{D}$ and $\dddot{D}$ on the paracontact distribution $\mathcal{H}=\ker(\eta)$ of $(\mathcal{M},\phi,\xi,\eta,g)$ are, respectively, the known P-connection, canonical connection and KT-connection on the corresponding Riemannian manifold equipped with traceless almost product structure (see e.g. [7,13]).

Using (3.1) and (3.3) regarding $\dddot{D}$, we get:

$$\dddot{T}(x,y,z)=\dddot{Q}(x,y,z)-\dddot{Q}(y,x,z).\tag{4.1}$$

For the covariant derivative of g with respect to $\dddot{D}$, bearing in mind (3.1) and (3.3) for $\dddot{D}$ and $\nabla g=0$, we have $(\dddot{D}_xg)(y,z)=-\dddot{Q}(x,y,z)-\dddot{Q}(x,z,y)$. It follows from the latter equality that the vanishing of $\dddot{D}g$ is equivalent to

$$\dddot{Q}(x,y,z)=-\dddot{Q}(x,z,y).\tag{4.2}$$

Since $\dddot{D}g=0$ holds, then (4.1) and (4.2) imply

$$\dddot{Q}(x,y,z)=\frac{1}{2}\left\{\dddot{T}(x,y,z)-\dddot{T}(y,z,x)+\dddot{T}(z,x,y)\right\}.$$

By virtue of the latter equality and the fact that $\dddot{T}$ is a 3-form, we obtain

$$\dddot{T}(x,y,z) = 2\dddot{Q}(x,y,z). \tag{4.3}$$

Therefore $\dddot{Q}$ is also a 3-form. Then, substituting (4.3) into (3.3) for $\dddot{D}$, we get that $\dddot{D}$ is defined by

$$g(\dddot{D}_x y, z) = g(\nabla_x y, z) + \frac{1}{2}\dddot{T}(x,y,z). \tag{4.4}$$

Let us denote by $\dddot{t}$, $\dddot{t}^*$ and $\widehat{\dddot{t}}$ the torsion forms of $\dddot{D}$, i.e. analogous definitions of (3.2) are valid for $\dddot{T}$. Since $\dddot{T}$ is a 3-form, it is obvious that the torsion forms vanish, i.e. $\dddot{t}(x) = \dddot{t}^*(x) = \widehat{\dddot{t}}(x) = 0$.

Lemma 4.2. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an arbitrary Riemannian Π-manifold. Then the following identity holds:*

$$N(x,y,z) = -\dddot{T}(x,y,z) + \dddot{T}(x,\phi y,\phi z) + \dddot{T}(\phi x,y,\phi z) - \dddot{T}(\phi x,\phi y,z).$$

Proof. Bearing in mind that $\dddot{D}$ is a natural connection, we use the form of F from Proposition 3.2 and substitute it into (2.7). Considering (4.3), we establish the truthfulness of the lemma. □

Theorem 4.3. *If $\dddot{D}$ exists on a Riemannian Π-manifold $(\mathcal{M},\phi,\xi,\eta,g)$, then ξ is a Killing vector field and $\mathfrak{S}\,F$ vanishes, i.e. $(\mathcal{M},\phi,\xi,\eta,g) \in \widehat{\mathcal{U}}_0$.*

Proof. Let us assume that $\dddot{D}$ exists. Then, using Proposition 3.2 and the fact that $\dddot{Q}$ is a 3-form, we prove that $\mathfrak{S}\,F(x,y,z) = 0$.

By virtue of Proposition 3.2 and (4.3), we obtain

$$F(x,\phi y,\xi) = -\frac{1}{2}\dddot{T}(x,y,\xi), \tag{4.5}$$

which by Lemma 2.1 implies

$$(\nabla_x\eta)(y) = \frac{1}{2}\dddot{T}(x,y,\xi). \tag{4.6}$$

Then, taking into account (4.6) and the fact that $\dddot{T}$ is a 3-form, we obtain that $(\mathfrak{L}_\xi g)(x,y) = 0$, i.e. ξ is a Killing vector field and according to Lemma 2.2 the theorem is proved. □

Theorem 4.4. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be a Riemannian Π-manifold belonging to the class $\widehat{\mathcal{U}}_0$. Then the third natural connection $\dddot{D}$ exists and it is determined by*

$$g(\dddot{D}_x y, z) = g(\nabla_x y, z) + \frac{1}{2}\dddot{T}(x,y,z).$$

The torsion tensor $\dddot{T}$ is expressed by

$$\dddot{T}(x,y,z) = \frac{1}{2}\,\mathfrak{S}\big\{F(x,y,\phi z) - 3\eta(x)F(y,\phi z,\xi)\big\} \tag{4.7}$$

or by the following equivalent shorter form

$$\dddot{T} = -\frac{1}{4}\mathfrak{S}\, N^{\mathrm{h}} + \eta \wedge \mathrm{d}\eta. \tag{4.8}$$

Proof. Firstly, let us assume that $(\mathcal{M},\phi,\xi,\eta,g) \in \widehat{\mathcal{U}}_0$, i.e., according to Lemma 2.2, ξ is a Killing vector field and $\mathfrak{S}\, F = 0$. Let us define a connection $\dddot{D}$ with a torsion tensor $\dddot{T}$ determined by (4.7).

We substitute y for ϕy into (4.7) and from the obtained result we subtract the form of (4.7) with substituted z for ϕz. Then, using (2.4) in the considered class, we get the following formula:

$$\dddot{T}(x,y,\phi z) - \dddot{T}(x,\phi y,z) = 2F(x,y,z). \tag{4.9}$$

The latter result implies that Proposition 3.2 holds for $\dddot{D}$ having the property

$$\dddot{T} = 2\dddot{Q}. \tag{4.10}$$

Using (4.7), (2.4) and Lemma 2.2, we obtain

$$\dddot{T}(x,y,z) + \dddot{T}(x,z,y) = -\frac{1}{2}\mathfrak{S}\, F(\phi x,\phi y,\phi z).$$

According to the latter equality, $\mathfrak{S}\, F = 0$ and (4.10), we have the second condition in Proposition 3.2 and that $\dddot{T}$ is a 3-form. Therefore, we proved that $\dddot{D}$ is the third natural connection.

Now, we prove the validity of the equivalent shorter form (4.8). Using (2.7), (2.4) and $\mathfrak{S}\, F = 0$, we compute the following

$$-\frac{1}{4}\mathfrak{S}\, N^{\mathrm{h}}(x,y,z) = \frac{1}{2}\mathfrak{S}\big\{F(x,y,\phi z) + \eta(x)F(y,\phi z,\xi)\big\}. \tag{4.11}$$

By (4.5), (4.6) and the fact that ξ is a Killing vector field, we express that $\mathrm{d}\eta(x,y) = -2F(x,\phi y,\xi)$ and therefore the following equality holds

$$(\eta \wedge \mathrm{d}\eta)(x,y,z) = -2\,\mathfrak{S}\,\eta(x)F(y,\phi z,\xi). \tag{4.12}$$

Summing (4.11) and (4.12), we get

$$\begin{aligned}-\frac{1}{4}\mathfrak{S}\, N^{\mathrm{h}}(x,y,z) + (\eta \wedge \mathrm{d}\eta)(x,y,z) &= \frac{1}{2}\mathfrak{S}\big\{F(x,y,\phi z) - 3\eta(x)F(y,\phi z,\xi)\big\}\\ &= \dddot{T}(x,y,z),\end{aligned}$$

which completes the proof of the theorem. □

Theorem 4.5. *The third natural connection $\dddot{D}$ exists on a Riemannian* Π-*manifold $(\mathcal{M},\phi,\xi,\eta,g)$ if and only if the associated Nijenhuis tensor $\widehat{N}$ vanishes on this manifold, i.e. $(\mathcal{M},\phi,\xi,\eta,g) \in \widehat{\mathcal{U}}_0$.*

Proof. The class $\widehat{\mathcal{U}}_0$ is characterised by $\widehat{N} = 0$. This fact together with Theorem 4.3 and Theorem 4.4 prove the truthfulness of the theorem. $\square$

The form of $\dddot{T}$ for $\widehat{\mathcal{U}}_0$ is given in (4.8). Using Lemma 2.1, (2.5), (2.6) and the following expressions of N obtained in [11]

$$\begin{aligned}\mathcal{F}_3: \quad & N(x,y,z) = -2\big\{F(\phi x,\phi y,\phi z) + F(\phi^2 x,\phi^2 y,\phi z)\big\},\\ \mathcal{F}_7: \quad & N(x,y,z) = 4F(x,\phi y,\xi)\eta(z),\end{aligned}$$

we get the form of $\dddot{T}$ in $\mathcal{F}_3$ and $\mathcal{F}_7$:

$$\begin{aligned}\mathcal{F}_3: \quad & \dddot{T} = -\frac{1}{4}\,\mathfrak{S}\,N^{\mathrm{h}},\\ \mathcal{F}_7: \quad & \dddot{T} = \eta\wedge \mathrm{d}\eta.\end{aligned} \tag{4.13}$$

As stated in Theorem 3.5, $\dot{D}$ and $\ddot{D}$ coincide if and only if $(\mathcal{M},\phi,\xi,\eta,g) \in \mathcal{U}_1$. Then, according to Theorem 4.5, we obtain that a necessary and sufficient condition for $\dot{D} \equiv \ddot{D}$ is that $\dddot{D}$ does not exist.

Also, let us remark that it is proved in [2] that the considered manifolds of the lowest dimension 3 belong to $\mathcal{F}_1\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{10}\oplus\mathcal{F}_{11}$. Therefore, it follows that in this case $\dddot{D}$ does not exist and $\dot{D}$ and $\ddot{D}$ coincide.

Theorem 4.6. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an arbitrary Riemannian Π-manifold belonging to the class $\widehat{\mathcal{U}}_0$. The first natural connection $\dot{D}$ is the average connection of the second natural connection $\ddot{D}$ and the third natural connection $\dddot{D}$, i.e.*

$$\dot{D} = \frac{1}{2}\left\{\ddot{D} + \dddot{D}\right\}.$$

Proof. Bearing in mind (3.4) and (4.13), we find that the identity $\dot{T} = \frac{1}{2}\left\{\ddot{T} + \dddot{T}\right\}$ holds for $\widehat{\mathcal{U}}_0$ (and in particular for $\mathcal{F}_3$ and $\mathcal{F}_7$). Since $\dot{D}$, $\ddot{D}$ and $\dddot{D}$ are natural connections and they preserve the metric, then they are completely determined by their torsion tensors. Therefore, using the latter equality, we proved the truthfulness of the theorem. $\square$

Considering Lemma 2.1, (2.3) and (4.7), we obtain the form of $\dddot{T}$ of type (1,2) as follows:

$$\begin{aligned}\dddot{T}(x,y) = \frac{1}{2}\big\{ & -2\,(\nabla_x\phi)\,\phi y + (\nabla_y\phi)\,\phi x - (\nabla_{\phi y}\phi)\,x\\ & + 3\eta(x)\nabla_y\xi - 4\eta(y)\nabla_x\xi + 2\,(\nabla_x\eta)\,(y)\,\xi\big\}.\end{aligned}$$

Corollary 4.7. *The third natural connection $\dddot{D}$ has the following properties:*

$$\begin{aligned}\dddot{T}(x,\phi y) &= \phi\dddot{T}(x,y) - 2\,(\nabla_x\phi)\,y,\\ \dddot{T}(\phi x,y) &= \phi\dddot{T}(x,y) + 2\,(\nabla_y\phi)\,x.\end{aligned}$$

Proof. By virtue of (3.1) for $\dddot{D}$ and (2.3), the first equality in the statement is equivalent to $\dddot{T}(x,\phi y,z) = \dddot{T}(x,y,\phi z) - 2F(x,y,z)$, which is a consequence of (4.9). The second one follows from the first and the fact that $\dddot{T}$ is a 3-form. $\square$

5. Example. Let us consider the following example given in [11]. Let $\mathcal{L}$ stand for a 5-dimensional Lie group with a basis $\{e_0, \dots, e_4\}$ of left-invariant vector fields on $\mathcal{L}$ and let the corresponding Lie algebra be determined by

$$\begin{aligned}
&[e_1, e_2] = [e_3, e_4] = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + 2\mu_1 e_0,\\
&[e_1, e_4] = -[e_2, e_3] = \lambda_3 e_1 + \lambda_4 e_2 + \lambda_1 e_3 + \lambda_2 e_4 + 2\mu_2 e_0,
\end{aligned}$$

where $\lambda_i, \mu_j \in \mathbb{R}$ $(i = 1, 2, 3, 4; j = 1, 2)$ and $[e_k, e_l] = 0$ in the other cases. We equip $\mathcal{L}$ with a Riemannian Π-structure (ϕ, ξ, η, g) as follows

$$\begin{aligned}
&\xi = e_0, \quad \phi e_1 = e_3, \quad \phi e_2 = e_4, \quad \phi e_3 = e_1, \quad \phi e_4 = e_2,\\
&\eta(e_0) = 1, \quad \eta(e_1) = \eta(e_2) = \eta(e_3) = \eta(e_4) = 0,\\
&g(e_i, e_i) = 1, \quad g(e_i, e_j) = 0, \quad i, j \in \{0, 1, \dots, 4\},\ i \neq j.
\end{aligned}$$

Then, the constructed manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ is a Riemannian Π-manifold. Moreover, in [11], it is proved that $(\mathcal{L}, \phi, \xi, \eta, g) \in \mathcal{F}_7$. Therefore, the considered manifold belongs to $\widehat{\mathcal{U}}_0$, i.e. bearing in mind Theorem 3.5 and Theorem 4.5, $\dot{D}$ and $\ddot{D}$ do not coincide and $\dddot{D}$ exists.

The following results are obtained in [11] for $\dot{D}$ and $\ddot{D}$ on $(\mathcal{L}, \phi, \xi, \eta, g)$:

$$\begin{aligned}
&\dot{T}_{102} = \dot{T}_{021} = \dot{T}_{304} = \dot{T}_{043} = \frac{1}{2}\dot{T}_{210} = \frac{1}{2}\dot{T}_{430} = \mu_1,\\
&\dot{T}_{104} = \dot{T}_{023} = \dot{T}_{302} = \dot{T}_{041} = \frac{1}{2}\dot{T}_{410} = \frac{1}{2}\dot{T}_{230} = \mu_2,\\
&\ddot{T}_{210} = \ddot{T}_{430} = 2\mu_1, \qquad \ddot{T}_{410} = \ddot{T}_{230} = 2\mu_2.
\end{aligned} \tag{5.1}$$

Let us consider $\dddot{D}$ on $(\mathcal{L}, \phi, \xi, \eta, g)$ such that it is defined by (4.4). Using (4.7) and the components $F_{ijk} = F(e_i, e_j, e_k)$ of F, given in [11],

$$F_{104} = F_{302} = -F_{203} = -F_{401} = \mu_1, \quad F_{102} = F_{304} = -F_{201} = -F_{403} = \mu_2,$$

we calculate the components $\dddot{T}_{ijk} = \dddot{T}(e_i, e_j, e_k)$ of $\dddot{T}$. The nonzero ones are defined by the following equalities, taking into account that $\dddot{T}$ is a 3-form:

$$\dddot{T}_{210} = \dddot{T}_{430} = 2\mu_1, \qquad \dddot{T}_{410} = \dddot{T}_{230} = 2\mu_2. \tag{5.2}$$

Let us also recall that since $\dddot{T}$ is a 3-form, its torsion forms are zero, i.e. $\dddot{t} = \dddot{t}^* = \hat{\dddot{t}} = 0$.

Taking into account (5.1) and (5.2), we obtain $\dot{T} = \frac{1}{2}\left\{\ddot{T} + \dddot{T}\right\}$. Since $\dot{D}$, $\ddot{D}$ and $\dddot{D}$ are completely determined by $\dot{T}$, $\ddot{T}$ and $\dddot{T}$, as we already remarked, then we have $\dot{D} = \frac{1}{2}\left\{\ddot{D} + \dddot{D}\right\}$, which is in unison with Theorem 4.6. Furthermore, the obtained results in (5.2) confirm the statements made in Corollary 4.7 for the case of $\mathcal{F}_7$.

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