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NOTES ON THE CO-PRIME ORDER GRAPH OF A GROUP

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Abstract

The co-prime order graph of a group G is the graph with vertex set G, and two distinct elements $x, y \in G$ are adjacent if $\gcd(o(x), o(y))$ is either 1 or a prime, where $o(x)$ and $o(y)$ are the orders of x and y, respectively. In this paper, we characterize finite groups whose co-prime order graphs are complete and classify finite groups whose co-prime order graphs are planar, which generalizes some results by Banerjee [3]. We also compute the vertex-connectivity of the co-prime order graph of a cyclic group, a dihedral group and a generalized quaternion group, which answers a question by Banerjee [3].

Key words: co-prime order graph, vertex-connectivity, group
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1. Introduction. Throughout this paper, all graphs are finite, undirected, with no loops and no multiple edges. Let Γ be a graph. Denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and the edge set of Γ, respectively. The *vertex-connectivity* of Γ, denoted $\kappa(\Gamma)$, is the minimum number of vertices of Γ such that the subgraph of Γ obtained by removing the vertices is disconnected or has only one vertex. An *embedding* of a graph into a surface is a drawing of the graph on the surface in such a way that its edges may intersect only at their endpoints. A graph is called

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planar if it can be embedded in the plane. Denote by K_n and $K_{m,n}$ the complete graph of order n and the complete bipartite graph, respectively.

All groups considered in this paper are finite. Let G be a group. An element of order 2 in G is called an *involution* of G. $o(g)$ denotes the order of an element $g \in G$. The cyclic group of order n is denoted by $\mathbb{Z}_n$. A *maximal cyclic subgroup* of G is a cyclic subgroup, which is not a proper subgroup of some cyclic subgroup of G. Denote by $\mathcal{M}_G$ the set of all maximal cyclic subgroups of G. A group is called a $\mathcal{P}$-group [4] if every non-trivial element of the group has prime order. For example, every elementary abelian p-group $\mathbb{Z}_p^n$ (namely, the n-fold direct product of $\mathbb{Z}_p$) is a $\mathcal{P}$-group, and the symmetric group S_3 on 3 letters is also a $\mathcal{P}$-group.

Graphs associated with groups and other algebraic structures have been actively investigated, because they have valuable applications (cf. [8,11]) and are related to automata theory (cf. [10,17]). The *power graph* of a group G has the vertex set G and two distinct elements are adjacent if one is a power of the other (cf. [1]). The *order supergraph* [7] of the power graph of G is a graph with vertex set G, and two distinct vertices x, y are adjacent if either $o(x)|o(y)$ or $o(y)|o(x)$. See [12,16] for some properties of this graph. The *order prime graph* (also called *co-prime graph*) of G has the vertex set G and two distinct elements x, y are adjacent if $\gcd(o(x), o(y)) = 1$, which was first introduced by Sattanathan and Kala [13]. Moreover, see [5,15] for some more properties of this graph. Motivated by the works on the power graphs, the order prime graphs and the order supergraphs of power graphs, Banerjee [3] introduced the *co-prime order graph* of a group G, which is denoted by $\Theta(G)$ and is the graph whose vertex set is G, and two distinct vertices x, y are adjacent if $\gcd(o(x), o(y))$ is either 1 or a prime. Note that the order prime graph of G is a spanning subgraph of $\Theta(G)$. Recently, Sehgal et al. [14] investigated the Laplacian spectrum of the co-prime order graph of G when G is a finite abelian p-group and a dihedral group.

In [3], Banerjee classified some special finite groups with complete co-prime order graphs and all cyclic groups whose co-prime order graphs are planar, and put forward the question: how to find all positive integers n such that $\kappa(\Theta(\mathbb{Z}_n))$ is the number of the set of all elements of order 1 or a prime?

Motivated by the results and the question in [3], in this paper we characterize all finite groups whose co-prime order graphs are complete and classify all finite groups whose co-prime order graphs are planar. We also give a necessary and sufficient condition for which $\kappa(\Theta(G))$ is equal to the number of the set of all elements of order 1 or a prime. As applications, we compute the vertex-connectivity of the co-prime order graph of a cyclic group, a dihedral group and a generalized quaternion group. Thus, we completely answer the above question.

2. Complete and planar co-prime order graphs. In [3], Banerjee classified some special finite groups with complete co-prime order graphs (such as, cyclic groups and dihedral groups) and classified all cyclic groups whose co-prime order graphs are planar (see [3], Theorems 3.3–3.9). In this section, we characterize all finite groups whose co-prime order graphs are complete (see Theorem 2.1), and classify all finite groups whose co-prime order graphs are planar (see Theorem 2.3).

In the following, we first give a characterization for the finite groups whose co-prime order graphs are complete, which implies some results of Banerjee in [3]. We always use G to denote a group with the identity element e.

Theorem 2.1. *$\Theta(G)$ is complete if and only if G is a $\mathcal{P}$-group.*

Proof. If g is an element of G with $o(g) \geq 3$, since $\gcd(o(g), o(g^{-1})) = o(g)$, then g and g^{-1} are non-adjacent in $\Theta(G)$ if and only if $o(g)$ is not prime, which proves the result. □

Remark 2.2. CHENG et al. [4] showed that the $\mathcal{P}$-groups are either p-groups of exponent p or Frobenius groups of particular type, or they are isomorphic to the alternating group on 5 letters. Therefore, Theorem 2.1 generalizes [3], Theorems 3.3–3.8.

We then classify all finite groups G such that $\Theta(G)$ is planar, which extends [3], Theorem 3.9.

Theorem 2.3. *$\Theta(G)$ is planar if and only if G is isomorphic to one of the following groups:*

(a) *the cyclic group $\mathbb{Z}_3$ of order 3;*

(b) *the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$;*

(c) *a cyclic 2-group;*

(d) *a generalized quaternion 2-group.*

We give some results before giving the proof of Theorem 2.3. For any group G, we use $\mathbf{P}_G$ to denote the set of all elements of order 1 or a prime in G, that is,

$$\mathbf{P}_G = \{x \in G : o(x) = 1 \text{ or } p, \text{ where } p \text{ is a prime}\}.$$

Observation 2.4. Let G be a group of order n and let $x \in G$. If $o(x)$ is 1 or a prime, then the degree of (x) is $n-1$ in $\Theta(G)$. In particular, $\mathbf{P}_G$ is a clique of $\Theta(G)$.

Theorem 2.5 (Kuratowski's Theorem)**.** *A graph is planar if and only if this graph does not contain K_5, $K_{3,3}$, or a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

Lemma 2.6. *If $\Theta(G)$ is planar, then G is a p-group, where p is a prime.*

Proof. Suppose for a contradiction that $|G|$ has two distinct prime divisors, say, p and q. Without loss of generality, we may assume $p < q$. It follows that $q \geq 3$, and so $|G| \geq 6$, which also implies $|\mathbf{P}_G| \geq 4$. Since $|G| \geq 6$, we may take an element w in $G \setminus \mathbf{P}_G$. Observation 2.4 implies that $\mathbf{P}_G \cup \{w\}$ is a clique of $\Theta(G)$. This means that $\Theta(G)$ has a subgraph isomorphic to K_5, contrary to Theorem 2.5. □

For $n \geq 2$, JOHNSON [9], pp. 44–45 defined the generalized quaternion group, which is denoted by Q_{4n} and has a presentation

$$Q_{4n} = \langle x, y : x^n = y^2, x^{2n} = e, y^{-1}xy = x^{-1}\rangle. \tag{1}$$

Remark that Q_{4n} has order $4n$ and has a unique involution x^n. Also, $o(x^i y) = 4$ for any $1 \leq i \leq 2n$,

$$Q_{4n} = \langle x\rangle \cup \{x^i y : 1 \leq i \leq 2n\}, \tag{2}$$

and

$$\mathcal{M}_{Q_{4n}} = \{\langle x\rangle, \langle xy\rangle, \langle x^2 y\rangle, \ldots, \langle x^n y\rangle\}, \quad x^n \in \bigcap_{M \in \mathcal{M}_{Q_{4n}}} M. \tag{3}$$

If n is a power of 2, then Q_{4n} is called a *generalized quaternion* 2*-group.*

Recall the following elementary result on p-groups.

Lemma 2.7 ([6], Theorem 5.4.10)**.** *Given a prime p, a p-group having a unique subgroup of order p is either a cyclic group or a generalized quaternion* 2*-group.*

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. We first prove the sufficiency. Clearly, $\Theta(\mathbb{Z}_3) \cong K_3$ and $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong K_4$, and so they all are planar. Next, we prove that $\Theta(\mathbb{Z}_{2^k})$ is planar, where $k \geq 1$. Note that $|\mathbf{P}_{\mathbb{Z}_{2^k}}| = 2$ and $\mathbb{Z}_{2^k} \setminus \mathbf{P}_{\mathbb{Z}_{2^k}}$ is an independent set of $\Theta(\mathbb{Z}_{2^k})$. It follows that $\Theta(\mathbb{Z}_{2^k}) \cong K_{1,1,2^k-2}$, which is planar (see also [3], Theorem 3.9). Finally, we prove that $\Theta(Q_{4\times 2^k})$ is planar, where $k \geq 1$. By (2) and (3), we have $\mathbf{P}_{Q_{4\times 2^k}} = \{a, e\}$, where a is the unique involution of $Q_{4\times 2^k}$. Furthermore, $Q_{4\times 2^k} \setminus \mathbf{P}_{Q_{4\times 2^k}}$ is an independent set of $\Theta(Q_{4\times 2^k})$, which implies that $\Theta(Q_{4\times 2^k}) \cong K_{1,1,2^{k+2}-2}$, and so is planar.

We next prove the necessity. Suppose that $\Theta(G)$ is planar. By Lemma 2.6, we have that G is a p-group. We cannot have $p \geq 5$ otherwise, Observation 2.4 would imply $\Theta(G)$ has a subgroup isomorphic to K_5 against Theorem 2.5. It follows that $p \leq 3$.

We now consider two cases.

Case 1. $p = 2$.

If $|G| \leq 4$, then G is isomorphic to one of $\mathbb{Z}_2$, $\mathbb{Z}_4$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$, as desired. In the following, we assume $|G| \geq 5$. Suppose for a contradiction that G has at least two distinct subgroups of order 2. Then in view of the fact that the number of involutions of a finite group of even order is odd (see [2], p. 18), it follows that G has at least three distinct involutions. This implies $|\mathbf{P}_G| \geq 4$. Since $|G| \geq 5$, we may choose an element w in $G \setminus \mathbf{P}_G$. By Observation 2.4, $\mathbf{P}_G \cup \{w\}$ is a clique of $\Theta(G)$, this contradicts Theorem 2.5. We conclude that G has a unique subgroup of order 2. It follows from Lemma 2.7 that G is either a cyclic group or a generalized quaternion 2-group, as desired.

Case 2. $p = 3$.

If G has at least two distinct subgroups of order 3, then $|\mathbf{P}_G| \geq 5$, and so $\Theta(G)$ has a subgraph isomorphic to K_5, which is impossible by Theorem 2.5. As a result, G has a unique subgroup of order 3. Now Lemma 2.7 implies that G is cyclic. Assume to the contrary that $|G| \geq 9$. Since G is cyclic, we have $|\mathbf{P}_G| = 3$. Take $x, y, z \in G \setminus \mathbf{P}_G$. Then the subgraph of $\Theta(G)$ induced by $\mathbf{P}_G \cup \{x, y, z\}$ has a subgraph isomorphic to $K_{3,3}$ by Observation 2.4, a contradiction by Theorem 2.5. We conclude that $G \cong \mathbb{Z}_3$, as desired. □

3. Vertex-connectivity. Recall that $\mathbf{P}_G$ is the set of all elements of order 1 or a prime in a group G. Banerjee [3] put forward the following question.

Question 3.1. ([3], Problem 4.5) Find all positive integers n such that $\kappa(\Theta(\mathbb{Z}_n)) = |\mathbf{P}_{\mathbb{Z}_n}|$.

Motivated by Question 3.1, in this section, we give a necessary and sufficient condition for $\kappa(\Theta(G)) = |\mathbf{P}_G|$. As applications, we compute the vertex-connectivity of the co-prime order graph of a cyclic group, a dihedral group and a generalized quaternion group. We also answer Question 3.1 (see Corollary 3.6).

Lemma 3.2. (i) *G is a $\mathcal{P}$-group if and only if $\kappa(\Theta(G)) = |G| - 1$.*
(ii) *If G is a non-$\mathcal{P}$-group, then $\kappa(\Theta(G)) \geq |\mathbf{P}_G|$.*

Proof. (i) Note that a graph of order n has vertex-connectivity $n - 1$ if and only if the graph is complete. Now the desired result follows from Theorem 2.1.

(ii) Suppose for a contradiction that $\kappa(\Theta(G)) < |\mathbf{P}_G|$. Then there exists a subset U of G such that $|U| < |\mathbf{P}_G|$ and the subgraph Δ of $\Theta(G)$ obtained by deleting the vertices in U is disconnected. Since $|U| < |\mathbf{P}_G|$, there exists $x \in \mathbf{P}_G$ such that $x \in V(\Delta)$. Now Observation 2.4 implies Δ is connected, a contradiction. □

Denote by $\pi_e(G)$ the set of orders of elements of G. Write

$$\nu(G) = \pi_e(G) \setminus \{o(x) : x \in \mathbf{P}_G\}.$$

Notice that $\nu(G) = \varnothing$ if and only if G is a $\mathcal{P}$-group, because G is a $\mathcal{P}$-group if and only if $\mathbf{P}_G = \pi_e(G)$. In the following, if G is not a $\mathcal{P}$-group, then we define a graph $\Gamma(G)$ which has the vertex set $\nu(G)$ and distinct vertices m, n are adjacent if $\gcd(m, n)$ is either 1 or a prime. By Lemma 3.2 (ii) and the definition of $\Gamma(G)$, one can easily get the following result.

Theorem 3.3. *Suppose that G is a non-$\mathcal{P}$-group. Then $\Gamma(G)$ is either disconnected or trivial if and only if $\kappa(\Theta(G)) = |\mathbf{P}_G|$.*

Using Lemma 3.2 (i) and Theorem 3.3, we characterize the vertex-connectivity of the co-prime order graph of an abelian group.

Theorem 3.4. *Let A be an abelian group of order n. Then*

$$\kappa(\Theta(A)) = \begin{cases} n-1, & \textit{if } A \textit{ is an elementary abelian } p\textit{-group;} \\ |\mathbf{P}_A|, & \textit{otherwise.} \end{cases}$$

Proof. By [4], A is a $\mathcal{P}$-group if and only if A is a p-group of exponent p, where p is a prime. Thus, A is a $\mathcal{P}$-group if and only if A is an elementary abelian p-group. Now Lemma 3.2 (i) implies that $\kappa(\Theta(A)) = n-1$ if and only if A is an elementary abelian p-group.

In the following, assume that A is not an elementary abelian p-group. Then A is a non-$\mathcal{P}$-group. By the fundamental theorem of finitely generated abelian groups, there exists an element $g \in A$ such that $o(a) \mid o(g)$ for any $a \in A$. Let $o(g) = m$. Then $m \in V(\Gamma(A))$. If $|V(\Gamma(A))| = 1$, then $\Gamma(A)$ is trivial. Otherwise, we may take $k \in V(\Gamma(A)) \setminus \{m\}$. It follows that $k \mid m$, and hence $\gcd(k, m) = k$. Since k is neither 1 nor a prime, we conclude that k and m is non-adjacent in $\Gamma(A)$, which implies that m is an isolated vertex. As a result, $\Gamma(A)$ is disconnected. Thus, in this case, we conclude that $\Gamma(A)$ is either disconnected or trivial. Thus, it follows from Theorem 3.3 that $\kappa(\Theta(A)) = |\mathbf{P}_A|$, as desired. □

By Theorem 3.4, we compute the vertex-connectivity of the co-prime order graph of a cyclic group.

Theorem 3.5. *Let n be a positive integer and let $n = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_t^{\lambda_t}$ be its canonical factorization. Then*

$$\kappa(\Theta(\mathbb{Z}_n)) = \begin{cases} n-1, & \textit{if } n \textit{ is a prime;} \\ 1 + \sum_{i=1}^{t} (p_i - 1), & \textit{otherwise.} \end{cases}$$

Proof. The required result follows from Theorem 3.4 and the fact that $\mathbb{Z}_n$ has a unique subgroup of order p_i for each $1 \le i \le t$, that is, $|\mathbf{P}_{\mathbb{Z}_n}| = 1 + \sum_{i=1}^{t} (p_i - 1)$. □

The following corollary of Theorem 3.5 is an answer to Question 3.1.

Corollary 3.6. *Let n be a positive integer. Then $\kappa(\Theta(\mathbb{Z}_n)) = |\mathbf{P}_{\mathbb{Z}_n}|$ if and only if n is a composite number.*

For $n \ge 3$, let D_{2n} denote the dihedral group of order $2n$. Namely,

$$D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle.$$

Remark that

$$D_{2n} = \langle a \rangle \cup \{ab, a^2 b, \dots, a^{n-1} b, b\}, \quad o(a^i b) = 2 \text{ for any } i \le 1 \le n-1. \tag{4}$$

and

$$\mathcal{M}_{D_{2n}} = \{\langle a\rangle, \langle ab\rangle, \langle a^2b\rangle, \ldots, \langle a^nb\rangle\}. \tag{5}$$

By (4) and (5), we have

$$\mathbf{P}_{D_{2n}} = \mathbf{P}_{\langle a\rangle} \cup \{ab, a^2b, \ldots, a^{n-1}b, b\}. \tag{6}$$

Combining Theorem 3.5 and (4)–(6), we have the following result, which computes the vertex-connectivity of the co-prime order graph of a dihedral group.

Corollary 3.7. *Let n be a positive integer and let $n = p_1^{\lambda_1}p_2^{\lambda_2}\ldots p_t^{\lambda_t}$ be its canonical factorization. Then*

$$\kappa(\Theta(D_{2n})) = \begin{cases} 2n-1, & \text{if } n \text{ is a prime;} \\ \sum_{i=1}^{t}(p_i-1)+n+1, & \text{otherwise.} \end{cases}$$

We conclude this section by computing the vertex-connectivity of the co-prime order graph of a generalized quaternion group.

Theorem 3.8. *Let n be a positive integer at least 2 and let $2n=2^{\lambda_0}p_1^{\lambda_1}p_2^{\lambda_2}\ldots p_t^{\lambda_t}$ be the canonical factorization of $2n$, where p_i is a prime for each $1 \le i \le t$. Suppose that Q_{4n} is the generalized quaternion group as presented in (1). Then*

$$\kappa(\Theta(Q_{4n})) = \begin{cases} 2+\sum_{i=1}^{t}(p_i-1), & \text{if } n \text{ is even;} \\ 2n, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Notice that Q_{4n} is a non-$\mathcal{P}$-group. Also, by (2), we have $\mathbf{P}_{Q_{4n}} = \mathbf{P}_{\langle x\rangle}$, which implies that $|\mathbf{P}_{Q_{4n}}| = 2 + \sum_{i=1}^{t}(p_i-1)$.

Suppose first that n is even. Then $\lambda_0 \ge 2$. Now we consider the graph $\Gamma(Q_{4n})$. It is clear that $2n \in \nu(Q_{4n})$. Moreover, let $m \in \nu(Q_{4n})$, (2) implies either $m = 4$ or $m \mid 2n$. Since n is even, we have that m must be a divisor of $2n$. Assume that $|\nu(Q_{4n})| \ge 2$. Then for any $l \in \nu(Q_{4n})\setminus\{2n\}$, we have $\gcd(2n, l) = l$. It follows that $2n$ is an isolated vertex in $\Gamma(Q_{4n})$ since l is a composite number. We conclude that $\Gamma(Q_{4n})$ is either disconnected or trivial. By Theorem 3.3, we have $\kappa(\Theta(Q_{4n})) = |\mathbf{P}_{Q_{4n}}|$, as desired.

In the following, suppose that n is odd. Then $\lambda_0 = 1$, and so $\gcd(2n, 4) = 2$. Now write

$$U = \{x^iy : 1 \le i \le 2n\}, \qquad V = \langle x\rangle \setminus \mathbf{P}_{\langle x\rangle}.$$

For each $g \in V$, since $o(g) \mid 2n$ and $\gcd(2n, 4) = 2$, we have $\gcd(o(g), 4) = 1$ or 2. It follows that in $\Theta(Q_{4n})$, every vertex in U is adjacent to every vertex in V.

Notice that $|U| = 2n$ and every element of U has order 4. Therefore, the subgraph of $\Theta(Q_{4n})$ obtained by deleting the vertices in $\langle x \rangle$ is disconnected, which implies $\kappa(\Theta(Q_{4n})) \leq 2n$. It suffices to show that Q_{4n} has no such subset W such that $|W| < 2n$ and the subgraph of $\Theta(Q_{4n})$ obtained by removing all vertices of W is disconnected.

Suppose for a contradiction that there exists $W \subseteq Q_{4n}$ such that $|W| < 2n$ and the subgraph of $\Theta(Q_{4n})$ obtained by deleting the vertices in W is disconnected. Then Observation 2.4 implies $\mathbf{P}_{Q_{4n}} \subseteq W$, that is, $\mathbf{P}_{\langle x \rangle} \subseteq W$. Also, since every vertex of U is adjacent to every vertex of V in $\Theta(Q_{4n})$, it follows that either $U \subseteq W$ or $V \subseteq W$. If $U \subseteq W$, then $|W| > 2n$, which is impossible. Consequently, we have $V \subseteq W$, and so $|W| \geq 2n$, a contradiction. □

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