

*In memory
of Acad. Stefan Dodunekov*

A FORMULA FOR THE n -TH PRIME NUMBER

Krassimir T. Atanassov

(Submitted on November 23, 2012)

Abstract

A new formula for the n -th prime number is introduced. It is based on an arithmetic function, similar to the operation “differentiation”, but defined only over natural numbers.

Key words: arithmetic function, prime number

2000 Mathematics Subject Classification: 11A25, 11A41

1. Introduction. Probably, the first explicit formula giving the n -th prime number was introduced in 1962 by L. VESHENEVSKIY in [1].

In PAULO RIBENBOIM's book [2], three other formulas for the n -th prime number are discussed, as they were introduced in 1964 by C. P. WILLANS [3], in 1971 by J. M. GANDHI [4] and also by J. Minač in an unpublished paper.

In [5,6], the author introduced four other explicit formulas, giving the n -th prime number.

Here, we will extend the list of these formulas with a new one.

Initially, let us define function sg by

$$sg(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases},$$

where x is a real number.

Below, we use the fact that every natural number n has a canonical representation in the form

$$n = \prod_{i=1}^k p_i^{\alpha_i},$$

where $p_1, \dots, p_k$ are different prime numbers, $\alpha_1, \dots, \alpha_k \geq 1$ are natural numbers, and $k \geq 1$.

2. Short remarks on an arithmetic function. In 1987, the author introduced an arithmetic function with properties similar to the operation “differentiation” [7,8]. Here, short remarks on this function are given and some of its properties are studied.

Let n have the above mentioned canonical form. Following [7,8], we define the function

$$(1) \quad \delta(n) = \sum_{i=1}^k \alpha_i p_1^{\alpha_1} \dots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i-1} p_{i+1}^{\alpha_{i+1}} \dots p_k^{\alpha_k}$$

It is similar to the operation “partial differentiation” with respect to the variable p_i , but for natural numbers only. Obviously, if p is a prime number, then from the definition it follows that

$$(2) \quad \delta(p) = 1.$$

From (1), it follows that for every natural number n

$$\delta(n) = \sum_{i=1}^k \alpha_i \frac{n}{p_i}.$$

Hence,

$$\delta(n) = n \cdot \sum_{i=1}^k \frac{\alpha_i}{p_i}.$$

Theorem 1 ([7,8]). *For every two natural numbers m and n , it holds that*

$$(a) \quad \delta(m.n) = \delta(m).n + m.\delta(n);$$

(b) *if $\frac{m}{n}$ is a natural number, then*

$$\delta\left(\frac{m}{n}\right) = \frac{\delta(m).n - m.\delta(n)}{n^2};$$

$$(c) \quad \delta(m^n) = n.m^{n-1}.\delta(m).$$

Now, it can be directly seen that for every natural number n

$$1 \leq \delta(n) < \infty$$

and for a non-prime number n

$$\delta(n) > 1.$$

For example,

$$\delta(p.q) = p + q \geq 5,$$

$$\delta(p^2) = 2p \geq 4,$$

where p and q are different prime numbers.

3. A new formula for p_n . It is well-known that function π determines the number of the prime numbers that are less than or equal to n (see, e.g., [2,9]) where $\pi(0) = 0$ and $\pi(1) = 0$.

Now, we use the definition of δ for constructing a new formula for the n -th prime number p_n .

Theorem 2. *The following equality holds for every natural number $n \geq 2$:*

$$(3) \quad \pi(n) = \sum_{k=2}^n \left[\frac{1}{\delta(k)} \right],$$

where $[x]$ is the integer part of the real number x .

Proof. Let $k \leq n$ be a natural number. If k is prime, then from (2)

$$\left[\frac{1}{\delta(k)} \right] = 1.$$

On the other hand, if k is not prime, then $\delta(k) > 1$, i.e.

$$\left[\frac{1}{\delta(k)} \right] = 0.$$

Therefore, the sum in the right-hand side of (3) is equal to $\pi(n)$.

Theorem 3. *For every natural number n*

$$(4) \quad p_n = \sum_{i=0}^{C(n)} sg(n - \pi(i)),$$

where (see [10], page 90, 5.27)

$$C(n) = \left[\frac{n^2 + 3n + 4}{4} \right].$$

Proof. Let us remind of the fact (see [10]) that

$$p_n < C(n)$$

for every natural number n . Now, we can see that for a fixed natural number n , the function $n - \pi(i)$ is monotonically decreasing in respect of i , and so is valid

for $sg(n - \pi(i))$. When $i = 0, 1, \dots, p_n - 1$, the values of $sg(n - \pi(i))$ are equal to 1, and when $i = p_n$, the expression $sg(n - \pi(i))$ is equal to 0.

Therefore, the sum in the right-hand side of (4) contains exactly p_n units.

Finally, from (3) and (4) we obtain

$$p_n = \sum_{i=0}^{C(n)} sg \left(n - \sum_{k=2}^i \left[\frac{1}{\delta(k)} \right] \right).$$

REFERENCES

- [1] VESHENEVSKIY L. Matematika v Shkole, 1962, No 5, 74–75 (in Russian).
- [2] RIBENBOIM P. The New Book of Prime Number Records, Springer, New York, 1995.
- [3] WILLANS C. The Mathematical Gazette, **48**, 1964, 413–415.
- [4] GANDHI J. Proc. Washington State Univ. Conf. on Number Theory, Washington State Univ., Pullman, 1971, 96–101.
- [5] ATANASSOV K. Compt. rend. Acad. bulg. Sci., **54**, 2001, No 7, 5–6.
- [6] ATANASSOV K. Notes on Number Theory and Discrete Mathematics, **15**, 2009, No 4, 23–27.
- [7] ATANASSOV K. Bulletin of Number Theory and Related Topics, **XI**, 1987, No 1, 3–26.
- [8] ATANASSOV K. Advanced Studies on Contemporary Mathematics, **8**, 2004, No 2, 177–182.
- [9] NAGELL T. Introduction to Number Theory, John Wiley & Sons, New York, 1950.
- [10] MITRINOVIĆ D., M. POPADIĆ. Inequalities in Number Theory. Niš, Univ. of Niš, 1978.

Department of Bioinformatics and Mathematical Modelling, IBPhBME
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105
1113 Sofia, Bulgaria
e-mail: krat@bas.bg