

ISOLATED GAPS IN ALMOST SYMMETRIC ARF
NUMERICAL SEMIGROUPS: CHARACTERIZATION
MODULO m CLASSES AND ENUMERATION

Meral Süer^{1✉}, Ravan Ertuğrul²

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Abstract

In this study, the isolated gaps of almost symmetric Arf numerical semigroups are systematically investigated. Taking the multiplicity m of the numerical semigroup as a reference, the isolated gaps are classified modulo m . For each isolated gap set I_i , an explicit description is given, its characterization is established, and its cardinality is computed. Consequently, new structural results concerning the gap structure of almost symmetric Arf numerical semigroups are obtained.

Key words: almost symmetric Arf numerical semigroups, isolated gap, multiplicity, numerical semigroups

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1. Introduction Let $\mathbb{N}_0 = \{1, 2, 3, \dots\} \cup \{0\}$. A subset $Y \subseteq \mathbb{N}_0$ is called a numerical set if $0 \in Y$ and its complement $\mathbb{N}_0 \setminus Y$ is finite. A numerical set Y is said to be proper if $Y \neq \mathbb{N}_0$. If a numerical set Y is also closed under addition (i.e., $x + y \in Y$ for all $y \in Y$), then Y is called a numerical semigroup.

Let Y be a numerical semigroup and $A \subseteq Y$. Then A is a generating set of Y if

$$Y = \langle A \rangle = \left\{ \sum_{k=1}^n y_n a_n : y_n \in A, a_n \in \mathbb{N}_0 \right\}.$$

If Y has no subset smaller than $A = \{y_1, y_2, \dots, y_n\}$ such that $y_1 < y_2 < \dots < y_n$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$, then A is called the minimal generating system of Y . The smallest non-zero element of the minimal generating set is called the multiplicity of Y and is denoted by $m(Y)$.

Let Y be a numerical semigroup. The set $\mathbb{N}_0 \setminus Y$ is called the gaps of Y and is denoted by $G(Y)$. The number of elements in the set of gaps is called the genus of Y and is denoted by $g(Y)$. The largest integer of the set of gaps is called the Frobenius number of Y and is denoted by $F(Y)$. (In particular, the set $\mathbb{N}_0$ is a semigroup, and $F(\mathbb{N}_0) = -1$.)

The elements of Y that are smaller than $F(Y)$ are called the small elements of Y . If a numerical semigroup has n small elements, it is customary to list them as $y_0 = 0 < y_1 < \dots < y_{n-1}$ and write

$$Y = \{y_0 = 0, y_1, \dots, y_{n-1}, F(Y) + 1 \rightarrow\},$$

the arrow at the end meaning that all subsequent integers belong to Y .

Let Y be a numerical semigroup. An integer $a \notin Y$ such that $a + b \in Y$ for all $b \in Y \setminus \{0\}$ is called a Pseudo-Frobenius number of Y , and the set of all Pseudo-Frobenius numbers of Y is denoted by $PF(Y)$. The number of elements in $PF(Y)$ is called the type of Y and is denoted by $t(Y)$.

A numerical semigroup Y is called an Arf numerical semigroup if, for all $x, y, z \in Y$ with $x \geq y \geq z$, the condition $x + y - z \in Y$ is satisfied [1]. In other words, a numerical semigroup Y satisfies this property if, for all $x, y \in Y$ with $x \geq y$, it holds that $2x - y \in Y$ [2].

Let Y be a numerical semigroup. Then Y is an almost symmetric semigroup if and only if the equality $g(Y) = \frac{F(Y) + t(Y)}{2}$ holds [3]. Let $PF(Y) = \{f_1 < f_2 < \dots < f_t = F(Y)\}$. In this case, a numerical semigroup Y is almost symmetric if and only if $f_i + f_{t-i} = F(Y)$ for all $i = 1, \dots, t-1$ [4].

Let N be a positive integer. A partition of N is a finite non-increasing sequence of positive integers $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$, satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \lambda_n$ and $\lambda_1 + \lambda_2 + \dots + \lambda_n = N$ (notation $\lambda \vdash N$). Each λ_i is called a part of the partition λ , and the integer n is called the length of the partition [5].

Every partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n] \vdash N$ can be represented by a Young diagram T_λ consisting of n columns of boxes with lengths $\lambda_1, \lambda_2, \dots, \lambda_n$. The correspondence $\lambda \rightarrow T_\lambda$ is a bijection between the set of partitions and the set of Young diagrams.

In a Young diagram, let u_i be the number of rows of length i for each $i = 1, 2, \dots, n$. Such a diagram is denoted by $T = 1^{u_1} 2^{u_2} \dots n^{u_n}$. The relationship between the partition parts and the sequence u_i is given by $\lambda_j = \sum_{i=j}^n u_i$. Consequently, $u_j = \lambda_j - \lambda_{j+1}$ for $1 \leq j < n$ and $u_n = \lambda_n$.

There is a bijection between the set of proper numerical sets and the set of Young diagrams. For a numerical set Y , the sequence $\{u_1, \dots, u_n\}$ is called its

Young sequence. Here, u_1 represents the number of gaps of Y less than the first non-zero element y_1 , and u_j represents the number of gaps between y_{j-1} and y_j .

Let Y be a proper numerical set with Young sequence $\{u_1, \dots, u_n\}$. Then Y is of the form:

$$Y = \left\{ 0, u_1 + 1, u_1 + u_2 + 2, \dots, \sum_{i=1}^n u_i + n, \rightarrow \right\} [6].$$

Example 1.1. Let us consider the partition $\lambda = [6, 2, 1] \vdash 9$. The sequence is computed as $u_1 = 6 - 2 = 4$, $u_2 = 2 - 1 = 1$, and $u_3 = 1$. Thus, the Young sequence is $\{4, 1, 1\}$. Using $y_j = \sum_{i=1}^n u_i + j$, we obtain:

- $y_1 = 4 + 1 = 5$
- $y_2 = (4 + 1) + 2 = 7$
- $y_3 = (4 + 1 + 1) + 3 = 9$.

The numerical set corresponding to the partition $\lambda = [6, 2, 1]$ is $Y = \{0, 5, 7, 9, \rightarrow\}$.

It is important to note that since every numerical semigroup is inherently a numerical set, all structural definitions provided above – developed through Young tableaux and partitions – remain valid for numerical semigroups as well. In this study, we specifically focus on numerical semigroups.

Let Y be a numerical semigroup and let $G(Y)$ denote its set of gaps. A gap $g \in G(Y)$, is called an isolated gap if $g - 1$ and $g + 1 \in Y$. The set of isolated gaps of Y is denoted by $I(Y)$. A numerical semigroup with no isolated gaps, that is, satisfying $I(Y) \neq \emptyset$, is called a perfect numerical semigroup. Suppose that $I(Y) \neq \emptyset$, and let $a \in Y \setminus \{0\}$. For each $i \in \{1, 2, \dots, a - 1\}$, define $I_i(Y) = \{x \in I(Y) : x \equiv i \pmod{a}\}$ [7].

In a 2019 study, MORENO-FRÍAS and ROSALES [8] introduced a new class of perfect numerical semigroups and investigated their properties, showing how these semigroups can be characterized, generated, and related to other known classes. In a subsequent 2020 study [9], they characterized perfect numerical semigroups of embedding dimension three and provided explicit formulas for their Frobenius numbers. In 2022, SMITH [7] constructed a new perfect numerical semigroup by adjoining the minimal isolated gap to a two-generated non-perfect numerical semigroup. SÜER and ERTUĞRUL [10] showed that in non-perfect Arf numerical semigroups, isolated gaps may combine to form a perfect Arf numerical semigroup, while certain proper subsets preserve the non-perfect structure.

In this study, almost symmetric Arf numerical semigroups satisfying $Y \neq \mathbb{N}_0$ are investigated. For this class of semigroups, which exhibits a specific structural form, the structure of the isolated gap set $I(Y)$ is examined in detail. The primary objective is to determine the precise structure of the sets $I(Y)$ and to compute the cardinalities of the subsets $I_i(Y)$, based on the relations introduced in Smith's

study [7]. As a result, it is shown that the set $I(Y)$ is finite and that its cardinality varies according to different parameter configurations.

2. Main results In this section, we examine the number of elements and the structure of the set $I_i(Y)$ of almost symmetric Arf numerical semigroups. The results show how these sets are classified depending on the structure of the semigroup.

Theorem 2.1 ([11]). *For some $j \geq 1$, the almost symmetric Arf semigroup corresponding to the partition $\lambda = [j]$ is of the form $Y = \{0, j+1, \rightarrow\}$.*

Theorem 2.2 ([11]). *For some $j \geq 1$ and $k \in \{1, 3, 5, \dots, 2j-1, \rightarrow\}$ the almost symmetric Arf semigroup corresponding to the partition $\lambda = [\alpha + \beta, \alpha, \alpha - 1, \dots, 2, 1]$ is of the form $Y = \{0, k+1, k+3, \dots, 2j+k+1, \rightarrow\}$.*

Theorem 2.3. *Let Y be a proper almost symmetric Arf numerical semigroup (that is, $Y \neq \mathbb{N}_0$). If $j \geq 1$ and $k \in \{1, 3, 5, \dots, 2j-1, \rightarrow\}$, then Y is of the form $Y = \{0, k+1, k+3, \dots, 2j+k+1, \rightarrow\}$.*

- i. *If $j = 1$ and $k \neq 1$, then $\#I(Y) = 1$. Moreover, we have $I_1(Y) = I(Y)$.*
- ii. *If $j \geq 1$ and $k = 1$, then $I(Y) = I_1(Y)$ and $\#I(Y) = j+1$.*
- iii. *If $j > 1$ ($j \geq 3$) and $1 < k < 2j-1$, then $\#I(Y) = j$. Let $j \equiv r \pmod{\frac{k+1}{2}}$ and $i \in \{1, 3, \dots, k\}$. Then*

$$\#I_i(Y) = \begin{cases} \frac{2(j-r)}{k+1} + 1, & \text{if } i \leq 2r-1 \\ \frac{2(j-r)}{k+1}, & \text{if } 2r-1 < i \leq k. \end{cases}$$

- iv. *If $j > 1$ and $k \geq 2j-1$, then $\#I(Y) = j$ and*

$$\#I_1(Y) = \#I_3(Y) = \dots = \#I_{(2j-1)}(Y) = 1.$$

Proof. Let Y be a proper almost symmetric Arf numerical semigroup with $Y \neq \mathbb{N}_0$. We begin the proof by taking into account the situations described in Theorem 2.1 and Theorem 2.2. Accordingly, we consider four distinct cases, depending on the possible values of the parameters k and j .

- i. *If $j = 1$ and $k \neq 1$, since $k+1 > 2$, we can write $Y = \{0, k+1, k+3, \rightarrow\}$ and hence $I(Y) = \{k+2\}$. Therefore, $\#I(Y) = 1$. Moreover, $m(Y) = k+1 = m$ and*

$$k+2 = (k+1) + 1 \equiv 1 \pmod{m}.$$

Thus, $I_1(Y) = \{k+2\}$. Since $I(Y) = \{k+2\}$, the equality $I(Y) = I_1(Y)$ holds.

- ii. If $j \geq 1$ and $k = 1$, then we can write $Y = \{0, 2, 4, \dots, 2 + 2j \rightarrow\}$ and $I(Y) = \{1, 3, \dots, 2j + 1\}$. Here, $\#I(Y) = \frac{2j+1-1}{2} + 1 = j + 1$. Therefore $\#I(Y) = j + 1$. Moreover, $m(Y) = 2 = m$ and we have

$$\begin{aligned} 1 &\equiv 1 \pmod{m}, \\ 3 &\equiv 1 \pmod{m}, \\ 5 &\equiv 1 \pmod{m}, \\ &\vdots \\ 2j + 1 &\equiv 1 \pmod{m}. \end{aligned}$$

Hence, $I_1(Y) = \{1, 3, \dots, 2j + 1\}$ and since every element of $I(Y)$ is congruent to 1 modulo $m = 2$, we get $I(Y) = I_1(Y)$.

- iii. If $j > 1$ and $1 < k < 2j - 1$, then $Y = \{0, k + 1, k + 3, \dots, k + (2j + 1) \rightarrow\}$. Since $k > 1$, we have $k + 1 > 2$; therefore, $1 \notin I(Y)$ and there is no isolated gap between 0 and $k + 1$ and

$$I(Y) = \{k + 2, k + 4, \dots, k + 2j\}.$$

Here $\#I(Y) = \frac{(k+2j)-(k+2)}{2} + 1 = j$. Moreover, $m(Y) = k + 1 = m$ and $\overline{k+1} = \{0, 1, \dots, k\}$. We have

$$I(Y) = \underbrace{\{(k+1) + 1, (k+1) + 3, \dots, (k+1) + (2j-1)\}}_{j \text{ times}}.$$

Since we are considering $I_i(Y)$, we have $i \leq k$ and i must be an odd integer (note that k is also odd by the definition of Y). Therefore, the set $\overline{k+1}$ contains $k + 1$ elements, of which $\frac{k+1}{2}$ are odd. Let $i \in \{1, 3, \dots, k\}$; there are $\frac{k+1}{2}$ elements. However, $I(Y)$ contains j elements. Since $k > 1$ and $k < 2j - 1$, we have $k + 1 > 2$ and $k + 1 < 2j$. So

$$2 < k + 1 < 2j \implies 1 < \frac{k+1}{2} < j.$$

Therefore, it is obvious that $\#I_i(Y) \geq 1$. Applying the division algorithm, we can write $j = t \cdot \left(\frac{k+1}{2}\right) + r$ from which it follows $t = \frac{2(j-r)}{k+1}$ and $j \equiv r \pmod{\frac{k+1}{2}}$

$$\begin{aligned} k + 2 &= (k + 1) + 1 \equiv 1 \pmod{m} \implies k + 2 \in I_1(Y) \\ k + 4 &= (k + 1) + 3 \equiv 3 \pmod{m} \implies k + 4 \in I_3(Y) \\ &\vdots \end{aligned}$$

$$k + r + 1 = (k + 1) + r \equiv r \pmod{m} \implies k + r + 1 \in I_r(Y)$$

$$\vdots$$

$$k + (k + 1) = (k + 1) + k \equiv k \pmod{m} \implies k + (k + 1) \in I_k(Y).$$

We have distributed j elements among the $\frac{k+1}{2}$ positions. Similarly,

$$k + (k + 1) + 2 = (k + 1) + (k + 1) + 1 \equiv 1 \pmod{m} \implies$$

$$k + (k + 1) + 2 \in I_1(Y)$$

$$k + (k + 1) + 4 = (k + 1) + (k + 1) + 3 \equiv 3 \pmod{m} \implies$$

$$k + (k + 1) + 4 \in I_3(Y)$$

$$\vdots$$

$$k + (k + 1) + r + 1 = (k + 1) + (k + 1) + r \equiv r \pmod{m} \implies$$

$$k + (k + 1) + r + 1 \in I_r(Y)$$

$$\vdots$$

$$k + (k + 1) + k + 1 = (k + 1) + (k + 1) + k \equiv k \pmod{m} \implies$$

$$k + (k + 1) + k + 1 \in I_k(Y).$$

We perform this process t times. That is, each $I_i(Y)$ contains t elements. After the last full cycle, r elements remain. These r elements are distributed among $r/2$ classes with odd indices. Thus, the final placement is as follows:

$$k + t(k + 1) + 2 = (t + 1)(k + 1) + 1 \equiv 1 \pmod{m} \implies$$

$$k + t(k + 1) + 2 \in I_1(Y)$$

$$k + t(k + 1) + 4 = (t + 1)(k + 1) + 3 \equiv 3 \pmod{m} \implies$$

$$k + t(k + 1) + 4 \in I_3(Y)$$

$$\vdots$$

$$\begin{aligned} k + 2j &= (k + 1) + (2j - 1) = (k + 1) + \left(2 \left(t \cdot \left(\frac{k + 1}{2} \right) + r \right) - 1 \right) \\ &= (1 + t)(k + 1) + (2r - 1) = 2r - 1 \pmod{m} \implies k + 2j \in I_{2r-1}(Y). \end{aligned}$$

It can be seen that

$$I_1(Y) = \{k + 2, k + (k + 1) + 2, \dots, k + t(k + 1) + 2\} \text{ and}$$

$$\#I_1(Y) = t + 1 = \frac{2(j - r)}{k + 1} + 1$$

$I_3(Y) = \{k+4, k+(k+1)+4, \dots, k+t(k+1)+4\}$ and

$$\#I_3(Y) = t+1 = \frac{2(j-r)}{k+1} + 1$$

$\vdots$

$I_{2r-1}(Y) = \{k+2r, k+(k+1)+2r, \dots, k+2j\}$ and

$$\#I_{2r-1}(Y) = t+1 = \frac{2(j-r)}{k+1} + 1$$

$I_{2r+1}(Y) = \{k+2r+2, k+(k+1)+2r+2, \dots, k+(t-1)(k+1)+2r+2\}$ and

$$\#I_{2r+1}(Y) = t = \frac{2(j-r)}{k+1}$$

$\vdots$

$I_k(Y) = \{k+(k+1), k+(k+1)+k+1, \dots, k+(t-1)(k+1)+k+1\}$ and

$$\#I_k(Y) = t = \frac{2(j-r)}{k+1}.$$

If $j > 1$ ($j \geq 3$) and $1 < k < 2j-1$, then $\#I(Y) = j$. Let $j \equiv r \pmod{\frac{k+1}{2}}$ and $i \in \{1, 3, \dots, k\}$. Then

$$\#I_i(Y) = \begin{cases} t+1, & \text{if } i \leq 2r-1 \\ t, & \text{if } 2r-1 < i \leq k \end{cases} = \begin{cases} \frac{2(j-r)}{k+1} + 1, & \text{if } i \leq 2r-1 \\ \frac{2(j-r)}{k+1}, & \text{if } 2r-1 < i \leq k. \end{cases}$$

iv. If $j > 1$ and $k \geq 2j-1$, then $k+1 \geq 2j > 2$ so $I(Y) = \{k+2, k+4, \dots, k+2j\}$. Therefore, $\#I(Y) = \frac{(k+2j)-(k+2)}{2} + 1 = j$. Hence, $\#I(Y) = j$. Since $k \geq 2j-1$ and $k \in \{1, 3, 5, \dots, 2j-1 \rightarrow\}$ we have $k+1 > 2j-1$. Additionally $m(Y) = k+1 = m$

$$k+2 = (k+1) + 1 \equiv 1 \pmod{m}$$

$$k+4 = (k+1) + 3 \equiv 3 \pmod{m}$$

$\vdots$

$$k+2j = (k+1) + (2j-1) \equiv 2j-1 < k+1 \pmod{m}.$$

Consequently, each element in $I(Y)$ belongs to exactly one $I_i(Y)$ set modulo m . Hence

$$\#I_1(Y) = \#I_3(Y) = \dots = \#I_{(2j-1)}(Y) = 1. \quad \square$$

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¹*Faculty of Sciences and Letters, Department of Mathematics, Batman University, Batman, Türkiye*
e-mail: meral.suer@batman.edu.tr (<https://orcid.org/0000-0002-5512-4305>)

²*Institute of Graduate Studies, Department of Mathematics, Batman University, Batman, Türkiye*
e-mail: ravanertugrul7200@gmail.com (<https://orcid.org/0009-0008-7324-7713>)