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ON THE SECURE DOMINATION NUMBER OF BIPARTITE GAP POSET GRAPHS

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Abstract

This article proposes a general approach to secure algebraically defined gap poset graphs of numerical semigroups using secure domination that has a significant proportion in error-tolerant networks. A secure dominating set D of a graph G is a subset of its vertex set, which is dominating with an additional property that, for all $v \notin D$, there is a vertex $u \in N(v) \cap D$ such that $(D \setminus \{u\}) \cup \{v\}$ is dominating. The secure domination number $\gamma_s(G)$ is the minimum size of secure dominating set of G. In this paper, the secure dominating set and the secure domination number for gap poset graphs of elementary numerical semigroups have been computed.

Key words: numerical semigroups, Frobenius number, gap poset graphs, secure domination number

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1. Introduction and preliminaries. Graph theory offers a strong configuration for representing interconnection networks as a simple graph $G = (V, E)$, where communication linkages are represented by the edge set E and network components are represented by the vertex set V. Computers, routers, and other network components are represented as vertices in interconnection networks, while the connections or communication links that connect them are represented as the

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graph's edges. In graphs, dominance is essential for safeguarding networks of connections. If every vertex in V is either contained in D or directly adjacent to at least one vertex in D, then $D \subseteq V$ of G is a dominating set [1]. The domination number $\gamma(G)$ is the minimum size of dominating set of G. Different domination versions have emerged to improve network resilience against node outages or attacks. One such variation is secure domination, which gives the network an extra layer of security by permitting the swap for damaged or attacked points with a different one instead of changing the dominating set requirement [2]. The minimum number of points needed to safeguard the network by supplying backup points is the secure domination number $\gamma_s(G)$. Cockayne et al. [3] introduced the concept of the secure domination number and examined its exact values for several fundamental graph classes, including paths, cycles, and complete multipartite graphs. Mynhardt et al. [4] employed a straightforward technique involving γ-excellent trees to identify the class of trees in which the domination number is equal to the secure domination number. Specifically, it is known that for a graph of order n the condition

$$\gamma_s(G) \leq 2\alpha(G),$$

is satisfied by the secure domination number, where $\alpha(G)$ is the graph's independence number [5]. A bound for a secure domination number of trees in terms of maximum degree was discovered by Cockayne in [6]. Grobler and Mynhardt [7] characterized a few classes of γ_s-ER critical graphs and investigated the effects of edge removal on $\gamma_s(G)$. Wang et al. [8] demonstrated that the decision version of the secure domination problem is NP-complete for doubly chordal graphs and star convex bipartite graphs. The necessary and sufficient conditions for connected graphs to have secure domination number equal to 2 are discussed in [9]. In particular, for any tree T with $n \geq 3$ vertices, ℓ leaves and t stems, the upper and lower bound on $\gamma_s(T)$ is

$$\frac{n+2}{3} \leq \gamma_s(T) \leq \frac{2n+2\ell-t}{4}$$

(See [10]). These advancements highlight the expanding scope of secure domination theory and encourage more research into secure domination parameters in algebraically defined graph families, like those resulting from numerical semigroups [11].

A submonoid is a non-empty subset η of $\mathbb{N}_0$ (the set of non-negative integers) that satisfies the following two conditions:
(i) $0 \in \eta$, and (ii) for all $a, b \in \eta$, the sum $a + b \in \eta$. A numerical semigroup η is a special type of submonoid $(\eta, +)$ of $\mathbb{N}_0$ for which the complement $\mathbb{N}_0 \setminus \eta$ is finite. The multiplicity of a numerical semigroup η is the smallest positive integer contained in η and is denoted by β. The elements belonging to the complement $\mathbb{N}_0 \setminus \eta$ are called the gaps of η. The Frobenius number of η, denoted by $F(\eta)$, is the largest element among these gaps. An integer $z \in \mathbb{Z}_0 \setminus \eta$ is called pseudo-Frobenius

number of η if $z + s \in \eta, \forall s \in \eta \setminus \{0\}$. The collection of all pseudo-Frobenius numbers is represented by $PF(\eta)$. The type of η is $t(\eta) = |PF(\eta)|$.

Let $\eta = \langle a_1, a_2, \ldots, a_d \rangle$, where $\gcd(a_1, a_2, \ldots, a_d) = 1$. If $\{a_1, a_2, \ldots, a_d\}$ has no proper subset that generates η, then $\{a_1, a_2, \ldots, a_d\}$ is called the minimal system of generators of η. The embedding dimension of η, represented as $e(\eta)$, refers to the number of elements in the minimal system of generators of η. A numerical semigroup η is said to be symmetric if $F(\eta)$ is odd and for each $a \in \mathbb{Z} \setminus \eta$, we have $F(\eta) - a \in \eta$. A numerical semigroup η is said to be pseudo-symmetric if $F(\eta)$ is even and for each $a \in \mathbb{Z} \setminus \eta$, either $F(\eta) - a \in \eta$ or $a = \dfrac{F(\eta)}{2}$.

2. Secure domination number of gap poset graphs of numerical semigroups. Let η be a numerical semigroup and G be its gap set. The order relation $\prec$ on G is defined by $x \prec y \iff y - x \in \eta$. It is straightforward to verify that $(G, \prec)$ is a poset. The simple graph associated with the gap poset G of a numerical semigroup η, denoted by $J_G(\eta)$, is defined as follows:

$$V\big(J_G(\eta)\big) = \{v_x : x \in G\},$$

and

$$E\big(J_G(\eta)\big) = \{v_x v_y \mid x \prec y \text{ or } y \prec x\}.$$

A numerical semigroup η satisfying $F(\eta) < 2\beta$ is called an elementary numerical semigroup. In this case $G = L \cup U$ with $L = \{1, 2, \ldots, \beta - 1\}$ and $U \subseteq \{\beta + 1, \beta + 2, \ldots, 2\beta - 1\}$. This gives $V\big(J_G(\eta)\big) = V_L \cup V_U$, where $V_L = \{v_\ell : \ell \in L\}$ and $V_U = \{v_u : u \in U\}$.

In this section, we compute the secure domination number for the gap poset graphs of elementary numerical semigroup $\eta \neq \mathbb{N}_0$.

Lemma 2.1 ([12, Lemma 4])**.** *Let η be a numerical semigroup. Then*

(i) $\mathrm{PF}(\eta) = \mathrm{Maximals}_{\leq_\eta}(\mathbb{Z} \setminus \eta)$,

(ii) *$g \in \mathbb{Z} \setminus \eta$ if and only if $f - g \in \eta$ for some $f \in \mathrm{PF}(\eta)$.*

Lemma 2.2 ([13, Lemma 48])**.** *Let η be a numerical semigroup. The following conditions are equivalent.*

(i) *η is symmetric.*

(ii) $\mathrm{PF}(\eta) = \{\mathrm{F}(\eta)\}$.

(iii) *If $n \in \eta$, then* $\mathrm{Max}_{\leq_\eta}\big(\mathrm{Ap}(\eta, n)\big) = \{\mathrm{F}(\eta) + n\}$.

(iv) $\mathrm{t}(\eta) = 1$.

Lemma 2.3. *Let $\eta \neq \mathbb{N}_0$ be a numerical semigroup, then $\gamma\big(J_G(\eta)\big) \leq \mathrm{t}(\eta)$.*

Proof. This follows from (ii) of Lemma 2.1. □

Theorem 2.4. *The domination number $\gamma(J_G(\eta)) = 1$ if and only if η is symmetric.*

Proof. ($\Rightarrow$) If $\gamma(J_G(\eta)) = 1$, then there exists some $g \in G$ such that $g - x \in \eta$ or $x - g \in \eta$, for every $x \in G$. Suppose $f_1 \in PF(\eta)$ with $f_1 \neq F(\eta)$ then from (i) of Lemma 2.1, it follows that $F(\eta) - f_1 \notin \eta$ and $f_1 - F(\eta) \notin \eta$. This implies that $g \notin PF(\eta)$. Note that $F(\eta) - g \in \eta$, and the vertex v_g is adjacent to v_{f_1}. We have the following two possibilities:

If $g - f_1 \in \eta$, then $F(\eta) - f_1 = (F(\eta) - g) + (g - f_1) \in \eta$, which is a contradiction.

Now, if $f_1 - g \in \eta$, then $F(\eta) - (f_1 - g) \in G$, otherwise $F(\eta) \in \eta$, which is false. Since $F(\eta) - g > f_1 - g$, therefore $F(\eta) - (f_1 - g) > g$. This implies that $g - (F(\eta) - (f_1 - g)) \notin \eta$ and $(F(\eta) - (f_1 - g)) - g = F(\eta) - f_1 \notin \eta$. This is again a contradiction, since the vertex v_g dominates every vertex of the graph $J_G(\eta)$. So our supposition that $|PF(\eta)| > 1$ is wrong, and therefore from Lemma 2.2, we get that η is symmetric.

($\Leftarrow$) If η is symmetric, then Lemma 2.3 gives $\gamma\big(J_G(\eta)\big) \leq 1$. Also, from Proposition 7 of [14], we have that $J_G(\eta)$ is connected and therefore $\gamma\big(J_G(\eta)\big) \geq 1$. Consequently, $\gamma\big(J_G(\eta)\big) = 1$. □

Proposition 2.5. *Let η be a symmetric elementary numerical semigroup with multiplicity β, then $\gamma_s(J_G(\eta)) = \beta - 1$.*

Proof. From Lemma 3 of [14], it follows that $J_G(\eta)$ is a bipartite graph and $V\big(J_G(\eta)\big) = V_L \cup V_U$, where $V_L = \{v_1, v_2, \ldots, v_{\beta-1}\}$, and $V_U \subseteq \{v_{\beta+1}, \ldots, v_{2\beta-1}\}$. Also, Proposition 7 of [14], gives that $J_G(\eta)$ is connected and $|J_G(\eta)| = \dfrac{F(\eta)+1}{2}$ $= \beta$, as $F(\eta) = 2\beta - 1$. This implies $V_U = \{v_{2\beta-1}\}$. Moreover, Theorem 2 of [15], implies that $J_G(\eta) \cong K_{1,\beta-1}$. Consequently, Proposition 10 of [3], gives $\gamma_s\big(J_G(\eta)\big) = \beta - 1$. □

Proposition 2.6. *Let $J_G(\eta)$ be a gap poset graph. Then the following hold:*

(1) *If the multiplicity of η is 2, then $\gamma_s\big(J_G(\eta))\big) = 1$.*

(2) *If $J_G(\eta)$ is a bipartite gap poset graph with order $2\beta - 3$ and $\beta + 1 \in \eta$, then* $\gamma_s\big(J_G(\eta))\big) = \left\lceil \dfrac{6\beta - 9}{7} \right\rceil$.

Proof. (1) : If the multiplicity of η is 2, then using Proposition 1 of [14], we obtain that $J_G(\eta)$ is a complete graph and therefore Proposition 9 of [3], gives $\gamma_s\big(J_G(\eta))\big) = 1$.

(2) : From Theorem 6 of [15], it follows that $J_G(\eta)$ is isomorphic to a path graph $P_{2\beta-3}$ and therefore Theorem 12 of [3], gives the required result. □

Theorem 2.7. *Let $J_G(\eta)$ be a bipartite gap poset graph of order $\beta + 1$, where $\beta \geq 4$. Then $J_G(\eta)$ is connected if and only if the gap set of η is $G = \{1, 2, \ldots, \beta - 1\} \cup \{2\beta - 1, 2\beta - k\}$, where $k \in \{2, 3, \ldots, \lceil \beta/2 \rceil\}$.*

Proof. The assumption $|V(J_G(\eta))| = \beta + 1$, implies that $V_U = \{v_{u_1}, v_{u_2}\}$, with $\beta + 1 \leq u_2 < u_1 \leq 2\beta - 1$.

($\Rightarrow$) Since $J_G(\eta)$ is connected, therefore, by using Lemma 1 of [15], we have $u_1 = F(\eta) = 2\beta - 1$. We may write $u_2 = 2\beta - k$ for some $k \in \{2, 3, \ldots, \beta - 1\}$. Now, for $v_\ell \in V_L$ we have

$$u_2 - \ell \in \eta \iff u_2 - \ell \geq \beta \iff \ell \leq \beta - k,$$

and

$$u_1 - \ell \in \eta \iff u_1 - \ell \neq u_2 \iff \ell \neq k - 1.$$

Thus, $N(v_{u_2}) = [v_1, v_{\beta-k}]$, and $N(v_{u_1}) = [v_1, v_{\beta-1}] \setminus \{v_{k-1}\}$. The connectivity of graph $J_G(\eta)$ forces $v_{k-1} \in N(v_{u_2})$, which gives

$$k - 1 \leq \beta - k \iff 2k \leq \beta + 1 \iff k \leq \frac{\beta + 1}{2}.$$

Since k is an integer, therefore $k \leq \lfloor \frac{\beta+1}{2} \rfloor$. Equivalently, $k \leq \lceil \frac{\beta}{2} \rceil$ and so we get $k \in \{2, 3, \ldots, \lceil \beta/2 \rceil\}$.

($\Leftarrow$) Note that $N(v_{u_2}) = [v_1, v_{\beta-k}]$, and $N(v_{u_1}) = [v_1, v_{\beta-1}] \setminus \{v_{k-1}\}$. Also, $k \leq \lceil \beta/2 \rceil \leq \frac{\beta+1}{2}$, gives $k - 1 \leq \beta - k$ and hence $N(v_{u_1}) \cap N(v_{u_2}) = [1, \beta - k] \setminus \{k - 1\} \neq \varnothing$. This implies every lower vertex v_ℓ is adjacent to either v_{u_1} or v_{u_2}, and v_{u_1}, v_{u_2} have a common neighbour. Hence any two vertices of the graph $J_G(\eta)$ are joined by a path of length at most 4, so we get that $J_G(\eta)$ is connected. □

Remark 2.8. If $J_G(\eta)$ is a connected bipartite gap poset graph having order $\beta + 1$, then $V\big(J_G(\eta)\big) = V_L \cup V_U$, where $V_L = \{v_1, v_2, \ldots, v_{\beta-1}\}$ and $V_U = \{v_{u_1} = v_{2\beta-1}, v_{u_2} = v_{2\beta-k}\}$. We decompose V_L such that $V_L = V_M \cup V_R \cup \{v_{k-1}\}$, where $V_M = N(v_{u_1}) \cap N(v_{u_2}) = \{v_1, v_2, \ldots, v_{\beta-k}\} \setminus \{v_{k-1}\} \neq \varnothing$, $V_R = \{v_{\beta-k+1}, \cdots, v_{\beta-1}\}$.

Lemma 2.9. *Let $J_G(\eta)$ be a connected bipartite gap poset graph of order $\beta + 1$, where $\beta \geq 4$. Then the set $D = V_U \cup V_R$ is a secure dominating set of $J_G(\eta)$.*

Proof. Note that $V(J_G(\eta)) \setminus D = V_M \cup \{v_{k-1}\} \subseteq N(v_{u_2})$, as $V_M \subseteq N(v_{u_2})$ by Remark 2.8 and the connectivity of graph $J_G(\eta)$ implies $k - 1 \leq \beta - k$. Since $v_{u_2} \in D$, so D is dominating. For the security condition, we have the following cases:

Case 1: If $v_x \in V_M = N(v_{u_1}) \cap N(v_{u_2})$, then choose v_{u_1} as a defender and define

$$D_1 = (D \setminus \{v_{u_1}\}) \cup \{v_x\} = \{v_{u_2}, v_x\} \cup V_R.$$

Since $V(J_G(\eta)) \setminus D_1 = \{v_{k-1}, v_{u_1}\} \cup V_M \setminus \{v_x\}$. Here, v_{u_1} is dominated by v_x and $V_M \setminus \{v_x\} \cup v_{k-1}$ are dominated by v_{u_2}. Consequently, D_1 dominates $J_G(\eta)$.

Case 2: If $v_x = v_{k-1}$, then we can choose v_{u_2} as a defender and consider

$$D_2 = (D \setminus \{v_{u_2}\}) \cup \{v_{k-1}\} = \{v_{u_1}, v_{k-1}\} \cup V_R.$$

Now we have $V(J_G(\eta)) \setminus D_2 = V_M \cup \{v_{u_2}\}$. Here, V_M is dominated by v_{u_1} and the vertex v_{u_2} is dominated by v_{k-1}. Thus D_2 dominates $J_G(\eta)$ and consequently, $D = V_U \cup V_R$ is a secure dominating set of $J_G(\eta)$. □

Proposition 2.10. *Let $J_G(\eta)$ be a connected bipartite gap poset graph of order $\beta + 1$, where $\beta \geq 4$. Then $\gamma_s\big(J_G(\eta)\big) = k + 1$.*

Proof. From Lemma 2.9, we have that $D = V_U \cup V_R$ is a secure dominating set of $J_G(\eta)$ and $|D| = |V_U| + |V_R| = 2 + (k-1) = k+1$. This gives $\gamma_s\big(J_G(\eta)\big) \leq k+1$. To prove the minimality of D, we need to show that for every secure dominating set D', one must have $|D'| \geq k+1$. Note that $|V_M| \neq 0$, and $N(v_m) = V_U$, for each $v_m \in V_M$. Also, for every $v_r \in V_R$, we have $N(v_r) = \{v_{u_1}\}$ and $N(v_{k-1}) = \{v_{u_2}\}$.

If $|D' \cap V_R| \leq |V_R| - 2$, then there exist two distinct leaves $v_{r_1}, v_{r_2} \in V_R \setminus D'$. For every $v_r \in V_R$, we have $N(v_r) = \{v_{u_1}\}$. By choosing v_{u_1} as a defender for v_{r_1}, then v_{r_2} is not dominated by resulting set, which contradicts the assumption that D' is secure. Hence,

$$|D' \cap V_R| \geq |V_R| - 1. \tag{1}$$

Case 1: If $v_{u_1} \notin D'$, then domination forces $V_R \subseteq D'$. Hence, $|D' \cap V_R| = k - 1$. Now, if $v_{u_2} \notin D'$, then since D' is dominating set therefore $V_M \subseteq D'$ and also D' contains v_{k-1} and hence $|D'| \geq k + 1$. Now suppose $v_{u_2} \in D'$.

- If $v_{k-1} \in D'$, then $D' \supseteq V_R \cup \{v_{u_2}, v_{k-1}\}$, so $|D'| \geq (k-1) + 2 = k + 1$.
- If $v_{k-1} \notin D'$, then $V_M \subseteq D'$. Suppose some $v_m \in V_M \setminus D'$, then its only possible defender is v_{u_2} but the exchange leaves v_{k-1} undominated. This contradicts security. So $V_M \subseteq D'$ and $|D'| \geq |V_R| + 1 + |V_M| \geq (k-1)+1+1 = k + 1$.

Case 2: Let $v_{u_2} \notin D'$, then domination forces $v_{k-1} \in D'$. Suppose $v_{u_1} \in D'$ (the case $v_{u_1} \notin D'$ is in Case 1). Using (1), $|D'| \geq |\{v_{u_1}, v_{k-1}\}| + |D' \cap V_R| \geq 2 + (k-2) = k$. If $|D'| \geq k + 1$ there is nothing to prove, so assume $|D'| = k$. Then $|D' \cap V_R| = k - 2$ and $D' \cap V_M = \varnothing$, hence $D' = \{v_{u_1}, v_{k-1}\} \cup (V_R \setminus \{v_r\})$ for exactly one leaf $v_r \in V_R$. The only possible defender of v_r is v_{u_1}, and the resulting set $V_R \cup \{v_{k-1}\}$ fails to dominate any $v_m \in V_M$. This contradicts the security of D'. Hence $|D'| \geq k + 1$.

Case 3: Let $V_U \subseteq D'$. By using (1), $|D' \cap V_R| \geq k - 2$, so $|D'| \geq |V_U| + |D' \cap V_R| \geq 2 + (k-2) = k$. If $|D'| \geq k + 1$ there is nothing to prove, so assume $|D'| = k$. Then $|D' \cap V_R| = k - 2$ and $|D' \cap (V_M \cup \{v_{k-1})| = 0$, hence $D' = V_U \cup (V_R \setminus \{v_r\})$ for exactly one leaf $v_r \in V_R$. Take $v_m \in V_M$ and its defender is v_{u_1} or v_{u_2}. If it is v_{u_1}, the resulting set $(D' \setminus \{v_{u_1}\}) \cup \{v_m\}$ fails to dominate v_r. If it is v_{u_2}, the resulting set $(D' \setminus \{v_{u_2}\}) \cup \{v_m\}$ fails to dominate v_{k-1}. Both contradict the security of D'. Hence $|D'| \geq k + 1$ and, consequently $\gamma_s\big(J_G(\eta)\big) = k + 1$. □

Example 2.11. Let η be elementary with $\beta = 7$ and $G = \{1, \ldots, 6, 11, 13\}$. Since $k = 3 \leq \lceil 7/2 \rceil$, so $J_G(\eta)$ is connected by Theorem 2.7. Here $u_1 = 13$, $u_2 = 11$ and

$$N(v_{11}) = \{v_1, v_2, v_3, v_4\}, \qquad N(v_{13}) = \{v_1, v_3, v_4, v_5, v_6\}.$$

Thus $V_M = \{v_1, v_3, v_4\}$, $v_{k-1} = v_2$ and $V_R = \{v_5, v_6\}$ (see Fig. 1).

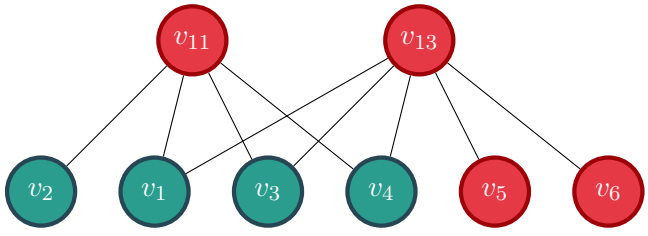

Fig. 1. $J_G(\eta)$ for $\beta = 7$, $k = 3$; red vertices form $D = V_U \cup V_R$

By Lemma 2.9, $D = \{v_{11}, v_{13}, v_5, v_6\}$ is a secure dominating set and by Proposition 2.10,

$$\gamma_s\big(J_G(\eta)\big) = k + 1 = 4.$$

3. Conclusion. In this paper, we have initiated a study of secure domination number of gap poset graphs of elementary numerical semigroups. We discovered explicit secure dominating sets and the accompanying secure domination numbers for elementary numerical semigroups by examining the structure of the gap set and the associated poset graph. Furthermore, these discoveries have practical applications, especially in the modelling of security and robustness in structured communication networks. In our upcoming research, we will investigate the related domination invariants, such as total, total secure, and Roman secure domination number of algebraically defined gap poset graphs.

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