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A NOTE ON MODULAR FORMS

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Abstract

For any integer $k \geq 2$ and $v \geq 1$, we can construct a kind of holomorphic functions defined on the upper complex plane, which satisfies the automorphy property of degree $2k$ for $SL_2(Z)$, and which is linearly independent with the classical Eisenstein series of weight $2k$.

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1. Introduction. Let $k \geq 2$ be an integer. According to Definitions 2 to 4 of §VII.2 of [1] (or the beginning of §III.2 of [2], which is the same), if the function $f(z)$ is analytic on the upper complex plane H, and for any

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z)$$

it satisfies (automorphy property)

$$f(Tz) = (J_T(z))^{-k} f(z), \text{ here } Tz = \frac{az+b}{cz+d}, \quad J_T(z) = (cz+d)^{-2}, \tag{1}$$

and

$$f(z) = \sum_0^\infty a_n q^n, \quad q = e^{2\pi i z}, \tag{2}$$

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(the series should be absolutely convergent), then $f(z)$ is called a holomorphic modular form of weight $2k$. Let M_k be the C-vector space of holomorphic modular forms of weight $2k$, and $\delta_k = \dim M_k$. Theorem 4(ii) of §VII.3 of [1] shows that

$$\delta_2 = \delta_3 = \delta_4 = \delta_5 = 1.$$

Let $\tilde{\delta}_k$ be the dimension of the C-vector space of holomorphic functions on H which only satisfy (1). In this paper we can show the following result.

Theorem 1. *For any $k \geq 2$ we have $\tilde{\delta}_k \geq 2$.*

For this purpose, we will construct a kind of new holomorphic functions on H which satisfy (1) for every positive integer $v \geq 1$ but may not satisfy (2), and we will show that $E_{2k}(z)$ and each $P_{v,k}(z)$ for $v \geq 1$ are linearly independent; here $E_{2k}(z)$ is the nomarlized Eisenstein series (see (2.12) of §III.2 of [2])

$$E_{2k}(z) = \frac{1}{2} \sum_{|c|+|d|\neq 0,\ (|c|,|d|)=1} (cz+d)^{-2k}, \quad \text{for any integer } k \geq 2. \tag{3}$$

Actually, our $P_{v,k}(z)$ for $v \geq 1$ comes from the appropriate generalization of $E_{2k}(z)$.

2. Proof of Theorem 1. For each positive integer $v \geq 1$, and each positive integer $k \geq 2$, let

$$P_{v,k}(z) = \sum_{|c|+|d|\neq 0,\ (|c|,|d|)=1} F_v(c,d;z)(J_T(z))^k.$$

Here the summation is taken for all integers c and d,

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z), \qquad Tz = \frac{az+b}{cz+d},$$

$$F_v(c,d;z) = e^{2\pi i v(Tz)},$$

the related a and b are any integers such that $ad-bc=1$ (from elementary number theory, such a and b always exist). We note that $P_{v,k}(z)$ has the same shape as the classical Eisenstein series of (3), except that $P_{v,k}(z)$ has an additional exponential factor $F_v(c,d;z)$. For $z = x+iy$ we have (see (1) of §VII.1 of [1])

$$L \equiv \operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{y}{|cz+d|^2} \Rightarrow |F_v(c,d;z)| = e^{-2\pi v L} < 1. \tag{4}$$

Thus in particular, when $y = \operatorname{Im}(z) \to +\infty$

$$F_v(c,d;z) \to 1 \ \text{ for } \ c \neq 0, \qquad F_v(c,d;z) \to 0 \ \text{ for } \ c = 0 \ (\text{then } |d| = 1). \tag{5}$$

From (4), by comparing with the absolutely convergent Eisenstein series of (3), we know that $P_{v,k}(z)$ is also absolutely convergent. But firstly we need to show

that each $F_v(c,d;z)$ is well-defined, that is, it depends only on c, d. To see this, for any given pair of integers $\{c,d\}$ satisfying

$$|c|+|d|\neq 0, \qquad (|c|,|d|)=1,$$

by a well-known fact of elementary number theory, we know that there exist integers a and b such that $ad-bc=1$, and if a' and b' also satisfy $a'd-b'c=1$, then there is an integer t such that

$$a'=a+tc, \qquad b'=b+td,$$

and thus

$$\frac{a'z+b'}{cz+d}=\frac{az+b}{cz+d}+t, \quad e^{2\pi iv(T(z))}=\ e^{2\pi iv(T'(z))}, \text{ here } \ T'z=\frac{a'z+b'}{cz+d},$$

which shows that $F_v(c,d;z)$ is well-defined. From this observation, by comparing with the shape of the Eisenstein series (as the classical modular form of weight $2k$), we can immediately conclude that $P_{v,k}(z)$ satisfies (1). Moreover $P_{v,k}(z)$ satisfies

$$\lim_{\operatorname{Im} z\to+\infty} P_{v,k}(z)=0, \tag{6}$$

because

$$P_{v,k}(z)=2F_v(0,1;z)+\sum_{|c|\neq 0,\ (|c|,|d|)=1} F_v(c,d;z)(J_T(z))^k,$$

and thus (4) and (5) give

$$\lim_{\operatorname{Im} z\to+\infty} F_v(0,1;z)=\lim_{y\to+\infty} e^{-2\pi vy}=0,$$

$$\lim_{\operatorname{Im} z\to+\infty} F_v(c,d;z)(J_T(z))^k=\lim_{y\to+\infty}(J_T(z))^k=0, \quad \text{for } \ c\neq 0.$$

The proof of Theorem 1 is then obvious, because for $k\geq 2$, if $\tilde{\delta}=1$, then $E_{2k}(z)$ and $P_{v,k}(z)$ are linearly dependent, then there are non-zero constants c_1 and c_2 such that

$$c_1E_{2k}(z)+\ c_2P_{v,k}(z)\equiv 0,$$

which is impossible, by letting $\operatorname{Im} z=y\to+\infty$ and using (6) and the well-known fact that

$$\lim_{\operatorname{Im} z\to+\infty} E_{2k}(z)=2.$$

Finally we give the direct verification that $P_{v,k}(z)$ satisfies (1). For any fixed integers A, B, C, D such that $AD-BC=1$, let

$$Sz\ =\ \frac{Az+B}{Cz+D}.$$

The equality $J_S(z)J_T(Sz) = J_{TS}(z)$ follows from

$$(Cz+D)\left(c\left(\frac{Az+B}{Cz+D}\right)+d\right) = ((cA+Cd)z+cB+Dd),$$

and

$$TS = \begin{pmatrix} aA+bC & aB+bD \\ cA+dC & cB+dD \end{pmatrix},$$

for any $T \in SL_2(Z)$ such that

$$Tz = \frac{az+b}{cz+d}.$$

Let $F_v(c,d,Sz) = e^{2\pi iv((TS)z)}$, we get

$$P_{v,k}(Sz) = \sum_{|c|+|d|\neq 0,\ (|c|,|d|)=1} F_v(c,d;Sz)(J_T(Sz))^k \tag{7}$$

$$= (J_S(z))^{-k}\left\{\sum_{|c|+|d|\neq 0,\ (|c|,|d|)=1} F_v(c,d;Sz)(J_{TS}(z))^k\right\}.$$

For the given S, using elementary number theory we can assert that

$$|c|+|d| \neq 0, \quad (|c|,|d|) = 1 \tag{8}$$

is equivalent to

$$|cA+Cd|+|cB+dD| \neq 0, \quad (|cA+Cd|,|cB+dD|) = 1.$$

For this we note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA+bC & aB+bD \\ cA+dC & cB+dD \end{pmatrix},$$

and thus $|c|+|d| \neq 0$ is equivalent to $|cA+Cd|+|cB+dD| \neq 0$, for if

$$|cA+Cd| = |cB+dD| = 0,$$

then

$$D(cA+Cd) = C(cB+dD) = B(cA+Cd) = A(cB+dD) = 0,$$

$$c = c(AD-BC) = D(cA+Cd) - C(cB+dD) = 0,$$

$$d = d(AD-BC) = A(cB+dD) - B(cA+Cd) = 0,$$

and vice versa. If

$$|c|+|d| \neq 0, \quad (|c|,|d|) = 1,$$

then $(|cA+Cd|, |cB+dD|) = 1$, for otherwise there is a positive integer $m > 1$ such that

$$cA + Cd = mt \quad \text{and} \quad cB + dD = ms$$

for integers t and s, then (here $U = aA + bC$ and $W = aB + bD$)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} U & W \\ mt & ms \end{pmatrix} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}, \quad c = m(tD - sC), \quad d = m(sA - tB),$$

is a contradiction to $(|c|, |d|) = 1$. Thus $(|cA+Cd|, |cB+dD|) = 1$. The converse is similar. Thus (8) is equivalent to

$$|cA + Cd| + |cB + dD| \neq 0, \quad (|cA + Cd|, |cB + dD|) = 1.$$

Then after the one to one substitution of variables

$$u = cA + dC, \quad w = cB + dD,$$

the summation for ordered integers $\{c, d\}$ satisfying (8) can be exchanged equivalently to a summation for ordered integers $\{u, w\}$ satisfying

$$|u| + |w| \neq 0, \quad (|u|, |w|) = 1,$$

and by using the definitions

$$(TS)z = T(Sz) = T\left(\frac{Az + B}{Cz + D}\right) = \frac{(aA + bC)z + aB + bD}{(cA + dC)z + cB + dD} = \frac{Uz + W}{uz + w},$$

we get

$$F_v(c, d; Sz) = e^{2\pi i v((TS)z)} = F_v(u, w; (TS)z),$$

and thus we find that

$$\begin{aligned}
(9) \qquad & \sum_{|c|+|d|\neq 0,\ (|c|,|d|)=1} F_v(c, d; Sz)(J_{TS}(z))^k \\
= & \sum_{\substack{|cA+Cd|+|cB+dD|\neq 0,\\ (|cA+Cd|,|cB+dD|)=1}} F_v(c, d; Sz)(J_{TS}(z))^k \\
= & \sum_{\substack{|cA+Cd|+|cB+dD|\neq 0,\\ (|cA+Cd|,|cB+dD|)=1}} F_v(cA + Cd, cB + dD; (TS)z)(J_{TS}(z))^k \\
= & \sum_{|u|+|w|\neq 0,\ (|u|,|w|)=1} F_v(u, w; \tilde{T}z)(J_{\tilde{T}}(z))^k = P_{v,k}(z);
\end{aligned}$$

here

$$\tilde{T}z = \frac{\tilde{a}z + \tilde{b}}{uz + w},$$

$\tilde{a}$ and $\tilde{b}$ are any integers such that $\tilde{a}w - \tilde{b}u = 1$. From (7) and (9) we see that $P_{v,k}(z)$ certainly satisfies (1).

REFERENCES

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