

DYNAMIC PROBLEM FOR A VISCOELASTIC HOLLOW
CYLINDER WITH COAXIAL ELASTIC INCLUSION

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Abstract

A problem of non-stationary longitudinal wave propagation in the cross-section of an infinite hollow linear-viscoelastic cylinder with coaxial elastic inclusion is considered. On the contact surfaces between the layers, continuity of displacements and stresses is assumed. The solution of the problem is constructed using the integral Laplace transform in time with subsequent reversal. For the case when each of the relaxation kernels is a finite sum of exponentials, the original is presented as a series of residues. Thus, the process of constructing a non-stationary solution is reduced to the search for the roots of the characteristic transcendental equation. The derived solution allowed to investigate the transient wave propagation in a viscoelastic cylinder with coaxial elastic inclusion with specified initial data.

Key words: viscoelastic dynamic problems, piecewise homogeneous solids, wave processes, layered cylinder

1. Introduction. Piecewise homogeneous deformable solids with viscoelastic components are widely used in modern technology, and they are also objects of

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study in geophysics, seismology, biology, etc. One of the important directions in the investigation of the behaviour of such structures under dynamic excitation is the development of analytical and semi-analytical methods for solving the initial and boundary value problems for piecewise homogeneous viscoelastic bodies. Results of studies employing both exact and approximate analytical methods were reported in the past decades [8–10, 13]. In recent years, attention has been paid to combining analytical methods with numerical ones [4, 5]. Among others, we would like to mention the boundary integral equations method [1], whose application to solve problems on the dynamic response of piecewise homogeneous viscoelastic media is discussed in [2, 3, 11]. Part of the recent publications has been devoted to harmonic waves in layered media with viscoelastic components [14, 15], as well as to the dynamics of viscoelastic layered thin-walled structures [6]. At the same time, despite the achievements in the field, there are still important issues that require attention and further investigation.

Among the variety of approaches to solve linear-viscoelastic dynamic problems, the most common is the integral Laplace transform technique, used alone or in combination with other methods. Due to the complexity of the inversion of the solution from the Laplace domain to time domain, it is often carried out asymptotically or with significant restrictions on the properties of the material. Earlier, the first author of this work established the general properties of the solution of non-stationary dynamic viscoelastic problems for piecewise homogeneous bodies in the Laplace domain [12]. Under certain conditions imposed on the initial data, based on the results [12], it is possible to reduce the construction of a solution of a non-stationary dynamic problem to finding the eigenvalues of the problem of free oscillations of a piecewise homogeneous viscoelastic body under consideration. The purpose of this work is to demonstrate such an approach to constructing solutions to non-stationary dynamic linear viscoelasticity problems for piecewise homogeneous bodies on a concrete example of a three-layer hollow cylinder. Here we also propose a method for finding the elements of the set of poles of the solution in the images in the vicinity of the limit points of this set.

2. Problem statement. Consider a problem of a non-stationary longitudinal wave propagation in a cross section of an infinite hollow viscoelastic cylinder with a coaxial elastic inclusion. Denote R_0 , R_3 – the inner and the outer radii of the cylinder, R_1 , R_2 – the inner and outer radii of the elastic inclusion ($R_0 < R_1 < R_2 < R_3$). The outer surface of the cylinder is free, while the inner surface is exposed to an evenly distributed radial load $Q(t)$ (Fig. 1). The cylinder is initially in a non-perturbed state.

Thus, the hollow cylinder consists of three coaxial layers, and continuity of displacements and radial stress is imposed at the layers' interfaces. This problem is a special case of the general one with an arbitrary number N coaxial viscoelastic layers, whose solution in Laplace domain is presented in [7]. The solution in the time domain is represented in a form containing an improper integral of a rapidly

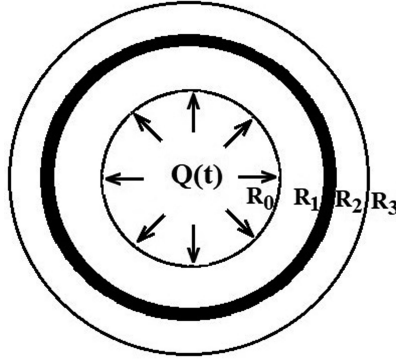


Fig. 1. Cross section of the three-layer cylinder

oscillatory function with complex parameter over the positive imaginary axis. In the present work, the solution of the problem in the Laplace domain for arbitrary N is significantly simplified. At the same time, for the case of hereditary kernels in the form of a finite sum of exponential functions, the solution in time domain is presented as a series of residues.

The following is the mathematical formulation of the problem for a cross section of the hollow cylinder with an arbitrary number of layers N and with continuity conditions at the layers' interfaces defined at $R = R_n$, $R_{n-1} < R_n < R_N$, $n = 1, 2, \dots, N-1$. The formulation is in the polar coordinate system (R, θ) whose origin coincides with the centre of the cross section of the cylinder. The following dimensionless variables are introduced:

$$\begin{aligned} r &= R/R_N, \quad \tau = t/t_0, \quad r_0 = R_0/R_N, \quad r_n = R_n/R_N, \quad r_N = 1, \\ Q_0\psi(\tau) &= Q(\tau t_0)/(2G_0^{(1)}), \quad \gamma_v^{(n)}(\tau) = t_0 T_v^{(n)}(\tau t_0), \quad \gamma_s^{(n)}(\tau) = t_0 T_s^{(n)}(\tau t_0), \\ \alpha_n &= c_N/c_n, \quad w_n = (1 - \nu_0^{(n)})/(1 - 2\nu_0^{(n)}), \quad u^{(n)}(r, \tau) = v^{(n)}(r R_N, \tau t_0)/R_N, \\ \sigma_r^{(n)}(r, \tau) &= P_R^{(n)}(r R_N, \tau t_0)/(2G_0^{(n)}), \quad \sigma_\theta^{(n)}(r, \tau) = P_\theta^{(n)}(r R_N, \tau t_0)/(2G_0^{(n)}), \end{aligned}$$

where $t_0 = R_N/c_N$; Q_0 is a dimensionless constant; $v^{(n)}(R, t)$ is the radial displacement, $P_R^{(n)}(R, t)$, $P_\theta^{(n)}(R, t)$ are radial and circumferential stresses, $T_v^{(n)}(t)$, $T_s^{(n)}(t)$ are volumetric and shear relaxation kernels, $G_0^{(n)}$, $\nu_0^{(n)}$ are the instantaneous shear modulus and Poisson's ratio, c_n is the longitudinal elastic wave velocity in the n -th layer.

The dimensionless formulation of the problem for $r_{n-1} \leq r \leq r_n$, $n = 1, 2, \dots, N$ reads:

$$(1) \quad \left(1 - d_1^{(n)}\right) \frac{\partial}{\partial r} \left[\frac{\partial u^{(n)}(r, \tau)}{\partial r} + \frac{u^{(n)}(r, \tau)}{r} \right] - \alpha_n^2 \frac{\partial^2 u^{(n)}(r, \tau)}{\partial \tau^2} = 0.$$

Herewith

$$\begin{aligned}
 \sigma_r^{(n)}(r, \tau) &= w_n \left(1 - \hat{d}_1^{(n)}\right) \frac{\partial u^{(n)}(r, \tau)}{\partial r} + (w_n - 1) \left(1 - \hat{d}_2^{(n)}\right) \frac{u^{(n)}(r, \tau)}{r}, \\
 \sigma_\theta^{(n)}(r, \tau) &= w_n \left(1 - \hat{d}_1^{(n)}\right) \frac{u^{(n)}(r, \tau)}{r} + (w_n - 1) \left(1 - \hat{d}_2^{(n)}\right) \frac{\partial u^{(n)}(r, \tau)}{\partial r}, \\
 (2) \quad \hat{d}_j^{(n)} \xi(r, \tau) &= \int_0^\tau d_j^{(n)}(\tau - \chi) \xi(r, \chi) d\chi, \quad j = 1, 2, \\
 d_1^{(n)}(\tau) &= \frac{1}{3(1 - \nu_0^{(n)})} \left[(1 + \nu_0^{(n)}) \gamma_v^{(n)}(\tau) + 2(1 - 2\nu_0^{(n)}) \gamma_s^{(n)}(\tau) \right], \\
 d_2^{(n)}(\tau) &= \frac{1}{3\nu_0^{(n)}} \left[(1 + \nu_0^{(n)}) \gamma_v^{(n)}(\tau) - (1 - 2\nu_0^{(n)}) \gamma_s^{(n)}(\tau) \right],
 \end{aligned}$$

and initial conditions:

$$(3) \quad u^{(n)}(r, 0) = 0, \quad \frac{\partial u^{(n)}}{\partial \tau}(r, 0) = 0.$$

The boundary conditions are as follows:

$$(4) \quad \sigma_r^{(1)}(r_0, \tau) = -Q_0 \psi(\tau), \quad \sigma_r^{(N)}(1, \tau) = 0, \quad \tau > 0.$$

The continuity conditions for $n = 1, \dots, N - 1$ interfaces read:

$$(5) \quad u^{(n)}(r_n, \tau) = u^{(n+1)}(r_n, \tau), \quad G_0^{(n)} \sigma_r^{(n)}(r_n, \tau) = G_0^{(n+1)} \sigma_r^{(n+1)}(r_n, \tau).$$

3. Solution method. Let us apply integral Laplace transform to equations (1), (2), (4), (5), taking into account the initial conditions (3), herewith $s \in \mathbb{C}$ and

$$U^{(n)}(r, s), \quad S_r^{(n)}(r, s), \quad S_\theta^{(n)}(r, s), \quad \Psi(s), \quad D_1^{(n)}(s), \quad D_2^{(n)}(s), \quad \Gamma_v^{(n)}(s), \quad \Gamma_s^{(n)}(s)$$

are correspondingly the transforms of functions

$$u^{(n)}(r, \tau), \quad \sigma_r^{(n)}(r, \tau), \quad \sigma_\theta^{(n)}(r, \tau), \quad \psi(\tau), \quad d_1^{(n)}(\tau), \quad d_2^{(n)}(\tau), \quad \gamma_v^{(n)}(\tau), \quad \gamma_s^{(n)}(\tau).$$

The solution for displacement in transforms takes the form ($n = 1, 2, \dots, N$):

$$(6) \quad U^{(n)}(r, s) = Q_0 \Psi(s) \frac{q_n(s)}{Z(s)} \left[I_1(y_n) g_1^{(n)}(s) + K_1(y_n) g_2^{(n)}(s) \right],$$

where $y_n(r, s) = r \alpha_n \beta_n(s)$, $r_{n-1} \leq r \leq r_n$,

$$\beta_n(s) = s \left(1 - D_1^{(n)}(s)\right)^{-1/2}, \quad Z(s) = - \left(T_1 g_1^{(1)} + T_2 g_2^{(1)}\right).$$

Functions $g_1^{(n)}(s)$ and $g_2^{(n)}(s)$ are determined from the recurrence relations:

$$\begin{pmatrix} g_1^{(n)} \\ g_2^{(n)} \end{pmatrix} = \begin{pmatrix} z_{11}^{(n)} & z_{12}^{(n)} \\ z_{21}^{(n)} & z_{22}^{(n)} \end{pmatrix} \begin{pmatrix} g_1^{(n+1)} \\ g_2^{(n+1)} \end{pmatrix}, \quad n = 1, 2, \dots, N-1,$$

where

$$\begin{aligned} g_1^{(N)} &= w_N(1 - D_1^{(N)})K_0(\beta_N) + (1 - \Gamma_s^{(N)})K_1(\beta_N)/\beta_N, \\ g_2^{(N)} &= w_N(1 - D_1^{(N)})I_0(\beta_N) - (1 - \Gamma_s^{(N)})I_1(\beta_N)/\beta_N, \\ z_{11}^{(n)} &= I_1(b_2^{(n)})\psi_{12}^{(n)} - K_1(b_1^{(n)})\psi_{21}^{(n)}, \\ z_{12}^{(n)} &= K_1(b_2^{(n)})\psi_{12}^{(n)} - K_1(b_1^{(n)})\psi_{22}^{(n)}, \\ z_{21}^{(n)} &= I_1(b_1^{(n)})\psi_{21}^{(n)} - I_1(b_2^{(n)})\psi_{11}^{(n)}, \\ z_{22}^{(n)} &= I_1(b_1^{(n)})\psi_{22}^{(n)} - K_1(b_2^{(n)})\psi_{11}^{(n)}, \end{aligned}$$

and also

$$\begin{aligned} \psi_{11}^{(n)} &= \frac{G_0^{(n)}\beta_n\alpha_n}{G_0^{(n+1)}\beta_{n+1}\alpha_{n+1}} \left[w_n(1 - D_1^{(n)})I_0(b_1^{(n)}) - (1 - \Gamma_s^{(n)})\frac{I_1(b_1^{(n)})}{b_1^{(n)}} \right], \\ \psi_{12}^{(n)} &= -\frac{G_0^{(n)}\beta_n\alpha_n}{G_0^{(n+1)}\beta_{n+1}\alpha_{n+1}} \left[w_n(1 - D_1^{(n)})K_0(b_1^{(n)}) + (1 - \Gamma_s^{(n)})\frac{K_1(b_1^{(n)})}{b_1^{(n)}} \right], \\ \psi_{21}^{(n)} &= w_{n+1}(1 - D_1^{(n+1)})I_0(b_2^{(n)}) - (1 - \Gamma_s^{(n+1)})I_1(b_2^{(n)})/b_2^{(n)}, \\ \psi_{22}^{(n)} &= -\left[w_{n+1}(1 - D_1^{(n+1)})K_0(b_2^{(n)}) + (1 - \Gamma_s^{(n+1)})K_1(b_2^{(n)})/b_2^{(n)} \right], \\ T_1 &= w_1(1 - D_1^{(1)})I_0(b_2^{(0)}) - (1 - \Gamma_s^{(1)})I_1(b_2^{(0)})/b_2^{(0)}, \\ T_2 &= -\left[w_1(1 - D_1^{(1)})K_0(b_2^{(0)}) + (1 - \Gamma_s^{(1)})K_1(b_2^{(0)})/b_2^{(0)} \right], \\ b_2^{(0)} &= r_0\alpha_1\beta_1, \quad b_1^{(n)} = r_n\alpha_n\beta_n, \quad b_2^{(n)} = r_n\alpha_{n+1}\beta_{n+1}. \end{aligned}$$

I_0 , I_1 , K_0 , K_1 are the modified Bessel functions of first and second kind. The values of q_n are determined with:

$$q_1 = 1/(\alpha_1\beta_1), \quad q_n = -\frac{G_0^{(n-1)}w_{n-1}(1 - D_1^{(n-1)})}{G_0^{(n)}r_{n-1}\beta_n\alpha_n}q_{n-1}, \quad n = 2, 3, \dots, N.$$

The transforms of stresses can be obtained using Eq. (6) and:

$$\begin{aligned} S_r^{(n)}(r, s) &= w_n(1 - D_1^{(n)})\frac{dU^{(n)}(r, s)}{dr} + (w_n - 1)(1 - D_2^{(n)})\frac{U^{(n)}(r, s)}{r}, \\ S_\theta^{(n)}(r, s) &= w_n(1 - D_1^{(n)})\frac{U^{(n)}(r, s)}{r} + (w_n - 1)(1 - D_2^{(n)})\frac{dU^{(n)}(r, s)}{dr}. \end{aligned} \quad (7)$$

From the general theorem proved in [12], it follows that if $\Psi(s)$, $\Gamma_v^{(n)}(s)$, $\Gamma_s^{(n)}(s)$ do not have branch points, then the transforms $U^{(n)}$, $S_r^{(n)}$, $S_\theta^{(n)}$ determined by formulas (6), (7) will not have branch points in the complex plane (despite the fact that they are expressed through functions having branch points).

Let the external load be defined as the Heaviside function $\psi(\tau) = h(\tau)$ and the relaxation kernels $\gamma_v^{(n)}$, $\gamma_s^{(n)}$ belong to the class of functions

$$(8) \quad \sum_{k=1}^K a_k e^{-b_k \tau}, \quad 0 \leq \sum_{k=1}^K a_k / b_k < 1,$$

where a_k , b_k are constants, $b_k > 0$ for all k . It means that their Laplace images have no branch points. Denote the set of zeros of the function $Z(s)$ as E_s . The poles of the functions $U^{(n)}$ are the elements of the set E_s and the point $s = 0$. It follows from the general results [12], that E_s is a set of eigenvalues of the problem of free oscillations of a layered cylinder ($\psi(\tau) \equiv 0$) occurring after sufficiently long time from the moment of their beginning that the nature of the oscillations does not depend on the method of their initiation. The limit points of the set E_s , along with the infinitely remote one, are the roots of the equations

$$(9) \quad 1 - D_1^{(n)}(s) = 0, \quad n = 1, 2, \dots, N.$$

We assume that all the roots of each of equations (9) are real and simple. In this case, after asymptotic investigation of the functions $U^{(n)}(r, s)$ in the vicinity of the limit points of the poles set, the originals $u^{(n)}(r, \tau)$ can be written in the form of an infinite sum of residues:

$$(10) \quad u^{(n)}(r, \tau) = u_0^{(n)}(r) + 2 \sum_{\{z_j\}} \Re \{ \text{Res}_{z_j} [U^{(n)}(r, s) e^{s\tau}] \} + \sum_{\{x_l\}} \text{Res}_{x_l} [U^{(n)}(r, s) e^{s\tau}],$$

$$u_0^{(n)}(r) = \text{Res}_{s=0} [U^{(n)}(r, s) e^{s\tau}].$$

Here $\{z_j\} \cup \{x_l\} = E_s$, z_j are complex, x_l are real ($j, l = 1, 2, 3, \dots$). With simple zeros of the function $Z(s)$ each residue in formula (10) in a point $s_k \in E_s$ is determined as:

$$\text{Res}_{s_k} [U^{(n)} e^{s\tau}] = \frac{Q_0 q_n(s_k)}{s_k Z'(s_k)} \left[I_1(y_n(r, s_k)) g_1^{(n)}(s_k) + K_1(y_n(r, s_k)) g_2^{(n)}(s_k) \right] e^{s_k \tau},$$

a prime in Z' denotes the derivative. It can be shown that $u_0^{(n)}(r)$ is a solution to a static elastic problem for a layered cylinder with long-term modules. Its expression is presented in [7].

Thus, for constructing the originals $u^{(n)}(r, \tau)$, the zeros of function $Z(s)$ have to be determined.

Let s_* be one of the roots of q -th equation (9). In this case $1 - D_1^{(q)}(s) \approx B(s - s_*)$ at $s \rightarrow s_*$, where B is a non-zero real constant (we assumed that all the roots of each of the equations (9) are real). Note that when s approaches s_* on the real axis so that the condition $B(s - s_*) < 0$ is met, $\beta_q(s)$ tends to infinity, remaining purely imaginary. This situation turns the functions included in the characteristic equation $Z(s) = 0$ in a small neighbourhood s_* into trigonometric ones. Let us assume as a hypothesis that the elements of the set E_s in a small neighbourhood of its limit point s_* are to be searched for respecting $\Re\{\beta_q(s)\} = 0$. Thus, provided that all the roots of equations (9) are real and simple, we propose to search for the elements of the set E_s in a small neighbourhood of each of these roots on the real axis.

Complex elements of the set E_s can be searched for using various iteration methods, taking purely imaginary elements of the spectral set for a linear elastic body as an initial approximation.

Let us return to the problem of a viscoelastic cylinder with the elastic inclusion (Fig. 1). In this problem $N = 3$, and since the materials of the first and third layers are the same, the dimensionless initial data are $r_0, r_1, r_2, \nu_0^{(2)}, \nu_0^{(3)}, G_0^{(3)}/G_0^{(2)}, \alpha_2 = c_3/c_2, \gamma_v^{(3)}(\tau), \gamma_s^{(3)}(\tau), \psi(\tau)$. Let $\psi(\tau) = h(\tau)$, the kernels $\gamma_v^{(3)}(\tau), \gamma_s^{(3)}(\tau)$ belong to class (8) and all the roots of every equation (9) be real and simple. The solution of the problem is presented in the form (10), taking into account the expressions (6). We will look for the zeros of the function $Z(s)$ in the way proposed above. Formulas for stresses $\sigma_r^{(n)}, \sigma_\theta^{(n)}$ are constructed similarly using $S_r^{(n)}, S_\theta^{(n)}$. For an arbitrary external load function $\psi(s)$ the solution will be based on the formulas (6), (7), (10) with a convolution.

4. Examples. We present the results of calculation of the parameters of a non-stationary wave process in the three-layer cylinder under consideration with specific initial data:

$$r_0 = 0.4, r_1 = 0.7, r_2 = 0.8, \nu_0^{(2)} = \nu_0^{(3)} = 0.3, G_0^{(3)}/G_0^{(2)} = 0.1, \alpha_2 = \sqrt{0.1}, \\ \gamma_v^{(3)}(\tau) \equiv 0, \gamma_s^{(3)}(\tau) = 0.6e^{-\tau}.$$

The limit point of the spectrum E_s found from the equation $1 - D_1^{(3)}(s) = 0$ will be only one: $s_* = -27/35$. In the process of calculations, we limited ourselves to five real components of the spectrum $x_1, x_2, \dots, x_5$. It is established that in the study of the non-stationary wave process, five real components of the spectrum are quite sufficient. The complex elements of the spectrum z_j were calculated using the Newton iteration method, while the corresponding purely imaginary values for the elastic cylinder were chosen as an initial approximation. It is established that in the study of the non-stationary wave process, a few dozen complex values z_j are sufficient.

We present the results for the case when the external load function is a smoothed Heaviside function $\psi(\tau) = 1 - e^{-100\tau}, \tau > 0$. Figures 2–4 show the

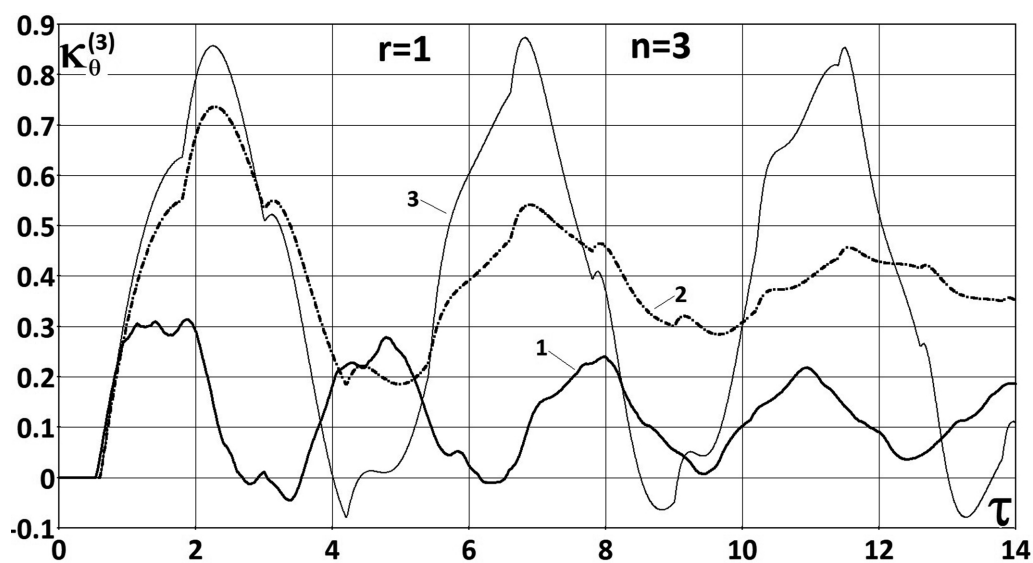


Fig. 2. Evolution of $\kappa_{\theta}^{(3)}$ in time at point $r = 1$. Curve 1 for viscoelastic three-layer cylinder, 2 – for viscoelastic homogeneous cylinder, 3 – for elastic homogeneous cylinder

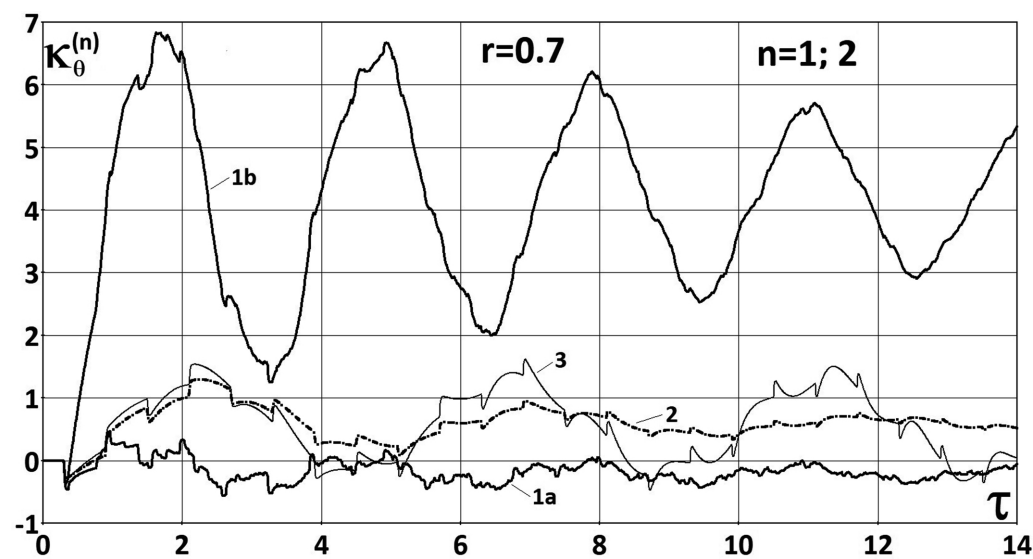


Fig. 3. Evolution of $\kappa_{\theta}^{(n)}$ in time at point $r = 0.7$. Curve 1a for $\kappa_{\theta}^{(1)}$; 1b – for $\kappa_{\theta}^{(2)}$; 2 – for viscoelastic homogeneous cylinder, 3 – for elastic homogeneous cylinder

evolution of the values of relative stresses:

$$\begin{aligned}\kappa_r^{(n)} &= G_0^{(n)} \sigma_r^{(n)} / \left(G_0^{(1)} Q_0 \right) = P_R^{(n)} / \left(2G_0^{(1)} Q_0 \right), \\ \kappa_{\theta}^{(n)} &= G_0^{(n)} \sigma_{\theta}^{(n)} / \left(G_0^{(1)} Q_0 \right) = P_{\theta}^{(n)} / \left(2G_0^{(1)} Q_0 \right)\end{aligned}$$

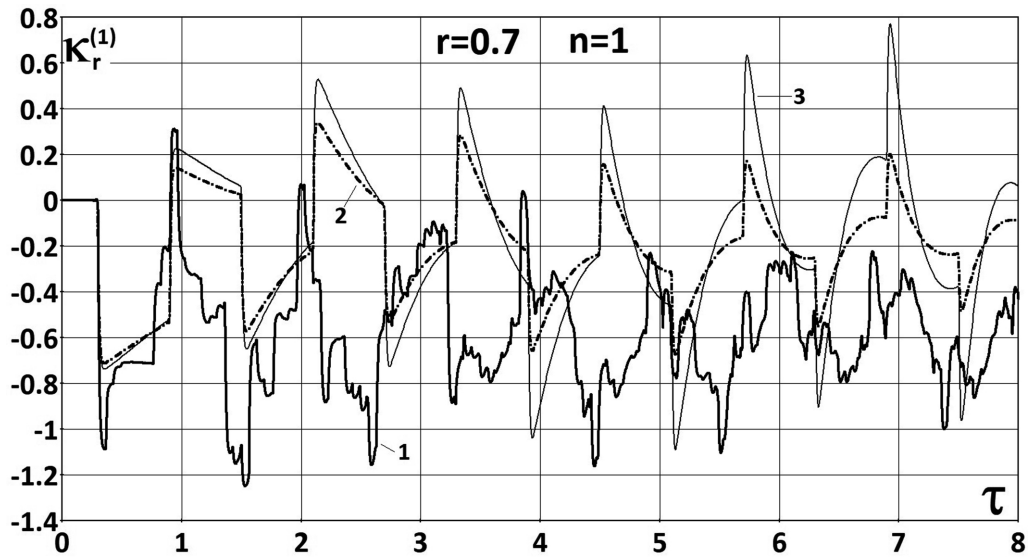


Fig. 4. Evolution of $\kappa_r^{(1)}$ in time at point $r = 0.7$. Curve 1 for viscoelastic three-layer cylinder, 2 – for viscoelastic homogeneous cylinder, 3 – for elastic homogeneous cylinder

in time at fixed points r . In Figure 2 curve 1 shows the evolutions $\kappa_\theta^{(3)}$ in time at point $r = 1$. In Figure 3 curve 1a and curve 1b show the evolution of $\kappa_\theta^{(1)}$ and $\kappa_\theta^{(2)}$, respectively, at point $r = 0.7$. In Figure 4 curve 1 shows the evolution of $\kappa_r^{(1)}$ at point $r = 0.7$. In Figures 2–4 curves 2 illustrate similar results for a viscoelastic cylinder without elastic inclusion (i.e. homogeneous) with initial data:

$$N = 1, \quad r_0 = 0.4, \quad \nu_0^{(1)} = 0.3, \quad \gamma_v^{(1)}(\tau) \equiv 0, \quad \gamma_s^{(1)}(\tau) = 0.6e^{-\tau}$$

and curves 3 show results for the same elastic homogeneous cylinder:

$$N = 1, \quad r_0 = 0.4, \quad \nu_0^{(1)} = 0.3, \quad \gamma_v^{(1)}(\tau) \equiv \gamma_s^{(1)}(\tau) \equiv 0.$$

Negative stresses are compressional. The results are obtained with 100 complex and the five real components of the spectrum for both viscoelastic cylinders.

It can be seen that a more rigid elastic inclusion significantly affects the behaviour of the circumferential stress. Compared to the case when the cylinder is homogeneous, this inclusion reduces the maximum amplitude of the circumferential stress at the outer boundary $r = 1$, as well as at the contact $r = 0.7$ from the side of a softer viscoelastic layer. Herewith, at the same contact from the side of a more rigid elastic material, the maximum amplitude of the circumferential stress becomes much greater. The elastic inclusion does not significantly affect the amplitude of the radial stress at the contact $r = 0.7$, but it changes the nature of the evolution of the radial stress over time.

5. Conclusion. The solution of a non-stationary dynamic problem for a linear viscoelastic hollow cylinder with coaxial elastic inclusion is derived and discussed. The problem is considered a special case of the problem for a cylinder with an arbitrary number of coaxial viscoelastic layers. The process of constructing a non-stationary dynamic solution is reduced to the search for the roots of the characteristic transcendental equation. A method of searching for these roots is proposed. The solution is valid for the entire time interval and does not require the assumption of low viscosity or constancy of the Poisson's ratio. The derived solution allowed to investigate the transient wave propagation in a cross section of a three-layer cylinder with specific initial data. With the help of calculations performed on the basis of the constructed solution, the effect of elastic inclusion on the evolution of stresses in time at selected locations of the cylinder cross section has been demonstrated.

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