Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 75, No 9, 2022

MATHEMATICS
Differential equations

INVERSE NODAL PROBLEM FOR p-LAPLACIAN STRING EQUATION WITH PRÜFER SUBSTITUTION

Emrah Yilmaz, Tuba Gulsen, Hikmet Koyunbakan

Received on November 1, 2021
Presented by V. Drensky, Member of BAS, on June 21, 2022

Abstract

We consider an inverse nodal problem for p-Laplacian string equation under some boundary conditions. Asymptotic formulas for eigenvalues and nodal parameters are constructed by modified Prüfer substitution. The most important process is to apply modified Prüfer substitution to get an exhaustive asymptotic estimate for eigenvalues. Moreover, a reconstruction formula for density function of p-Laplacian string equation is obtained by nodal parameters. Generated outcomes are the generalization of the known string problem.

Key words: inverse nodal problem, Prüfer substitution, p-Laplacian string equation

2020 Mathematics Subject Classification: 34A55, 34L05, 34L20

1. Introduction. Consider the following p-Laplacian string equation

$$-\left(y'^{(p-1)}\right)' = (p-1)\,\lambda\rho(x)y^{(p-1)},\ 0 < x < 1, \tag{1.1}$$

with the boundary conditions

$$y(0) = y(1) = 0, \tag{1.2}$$

where $p > 1$, λ is a spectral parameter and $y^{(p-1)} = |y|^{(p-2)}\,y$. During this study, we need to suppose that $\rho(x)$ is a positive C^2-function defined on $[0, 1]$ and $y(x, \lambda)$ denotes the solution of (1.1)–(1.2). It should be emphasized that (1.1) turns to

DOI:10.7546/CRABS.2022.09.02

classical string equation when $p = 2$ which is studied by several authors (see [13, 18]).

The determination of differential operators by spectral datas has become an interesting problem recently. The most important of these operators is Sturm–Liouville (SL) operator. In inverse SL problem, one tries to recover both potential function and constants by the eigenvalues, norming constants and spectral function. McLaughlin [11] suggested a novel and effective method which is related to nodal points to construct SL operator in 1988. This effective technique is called inverse nodal problem. Independently, Shen [13] examined the link between the nodal parameters and the density function of the string equation in the same year. Inverse nodal problem has been considered and various reconstruction formulas have been obtained and examined for different operators by many authors (see [7, 8, 10, 16, 19]).

The set $X_n = \left\{x_j^n\right\}_{j=1}^{n-1}$ is known as the set of nodal points for eigenfunction $y_n(x, \lambda_n)$ corresponding to λ_n for $\forall n \in \mathbb{N}$ and, $l_j^n = x_{j+1}^n - x_j^n$ is the corresponding nodal length. $y_n(x, \lambda_n)$ has exactly $n-1$ nodal points as $0 = x_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < x_n^{(n)} = 1$ on $(0,1)$. To say something on inverse nodal problem for p-Laplacian string equation, we should firstly recall the generalized sine function S_p which is the solution of the problem

$$-\left(S_p'^{(p-1)}\right)' = (p-1)S_p^{(p-1)}, \tag{1.3}$$

$$S_p(0) = 0,\ S_p'(0) = 1,$$

where S_p and S_p' are periodic which satisfy the relation

$$|S_p(x)|^p + \left|S_p'(x)\right|^p = 1,$$

for any $x \in \mathbb{R}$. These functions are the p-generalizations of classical sine and cosine. Here,

$$\pi_p = \int_0^1 \frac{2}{(1-t^p)^{\frac{1}{p}}} dt = \frac{2\pi}{p \sin\left(\frac{\pi}{p}\right)},$$

is the first zero of S_p in positive axis (see [3]). Presently, we will bring to mind some basic properties of S_p.

Lemma 1.1 ([3]). *Let S_p be the generalized sine function. Then, the following relations hold.*

i) *For $S_p' \neq 0$,*

$$\left(S_p'\right)' = -\left|\frac{S_p}{S_p'}\right|^{p-2} S_p.$$

ii)

$$\left(S_p S_p'^{(p-1)}\right)' = \left|S_p'\right|^p - (p-1)S_p^p = 1 - p\left|S_p\right|^p = (1-p) + p\left|S_p'\right|^p.$$

p-Laplacian eigenvalue problems are extremely important to get more general results in spectral theory. These problems have been considered by several authors (see [1,4–6,9,12,14,15]).

This study is arranged as follows. In Section 2, we construct some asymptotic formulas for eigenvalues and nodal parameters of the problem (1.1)–(1.2) by using Prüfer substitution. In Section 3, we obtain a reconstruction formula for density function of (1.1)–(1.2). Eventually, we give a summary for this study with a conclusion in Section 4.

2. Some asymptotic estimates for p-Laplacian string equation. Here, we firstly obtain asymptotic expansion of eigenvalues for (1.1) p-Laplacian string equation with (1.2) boundary conditions. For this purpose, we define a modified Prüfer substitution which is one of the strongest methods to study the solutions of a self adjoint 2nd order linear differential equation as

$$\begin{aligned} \lambda^{1/p}\rho^{1/p}(x)y(x) &= R(x)S_p\left(\lambda^{1/p}\ \theta(x,\lambda)\right), \\ y'(x) &= R(x)S_p'\left(\lambda^{1/p}\ \theta(x,\lambda)\right), \end{aligned} \tag{2.1}$$

or

$$\frac{y'(x)}{y(x)} = \lambda^{1/p}\rho^{1/p}(x)\frac{S_p'\left(\lambda^{1/p}\ \theta(x,\lambda)\right)}{S_p\left(\lambda^{1/p}\ \theta(x,\lambda)\right)}, \tag{2.2}$$

where $R(x)$ is amplitude and $\theta(x)$ is Prüfer variable [17]. Differentiating both sides of Eq. (2.2) with respect to x and considering Lemma 1.1, we obtain

$$\begin{aligned} \theta'(x,\lambda) = \rho^{1/p}(x) &+ \frac{\rho'(x)}{p\lambda^{1/p}\rho(x)}\frac{S_p\left(\lambda^{1/p}\ \theta(x,\lambda)\right)}{S_p'\left(\lambda^{1/p}\ \theta(x,\lambda)\right)} \\ &- \frac{\rho'(x)}{p\lambda^{1/p}\rho(x)}\frac{S_p\left(\lambda^{1/p}\ \theta(x,\lambda)\right)}{S_p'\left(\lambda^{1/p}\ \theta(x,\lambda)\right)}S_p^p\left(\lambda^{1/p}\ \theta(x,\lambda)\right). \end{aligned} \tag{2.3}$$

This relation will play an important role in the proofs of Theorems 2.1, 2.2 and 2.3. In addition, another equality which is very important and known as Riemann–Lebesgue lemma for our proofs is given below.

Lemma 2.1 ([15]). *Define $\theta(x,\lambda_n)$ as in (2.1) and $\phi_n(x) = S_p^p\left(\lambda_n^{1/p}\theta(x,\lambda_n)\right) - \dfrac{1}{p}$. Then, for any $g \in L^1(0,1)$*

$$\int_0^1 \phi_n(x)g(x)dx = 0.$$

Therefore, we can construct asymptotic estimates of eigenvalues and nodals for the problem (1.1), (1.2).

Theorem 2.1. *The eigenvalues of the problem (1.1), (1.2) are in the form of*

$$\lambda_n^{1/p} = \frac{n\pi_p}{c_\rho(1)} + \frac{(p-1)}{p^2 c_\rho(1)}\int_0^1 \frac{\rho'(x)S_p\left(\lambda_n^{1/p}\ \theta(x,\lambda_n)\right)}{\rho(x)S_p'\left(\lambda_n^{1/p}\ \theta(x,\lambda_n)\right)}dx + O\left(\frac{1}{n^{p-2}}\right) \tag{2.4}$$

as $n\to\infty$ where $c_\rho(1) = \int_0^1 \rho^{1/p}(x)dx$.

Proof. First of all, we need to integrate both sides of (2.3) from 0 to 1 with respect to x to get

$$\theta(1,\lambda) = \int_0^1 \rho^{1/p}(x)dx + \frac{1}{p\lambda^{1/p}}\int_0^1 \frac{\rho'(x)S_p}{\rho(x)S_p'}dx - \frac{1}{p\lambda^{1/p}}\int_0^1 \frac{\rho'(x)S_p}{\rho(x)S_p'}S_p^p dx,$$

where $\theta(0,\lambda_n) = 0$ and $\theta(1,\lambda_n) = \dfrac{n\pi_p}{\lambda_n^{1/p}}$. Here, we suppose that λ_n is an eigenvalue of the problem (1.1)–(1.2). By Lemma 2.1, we know that

$$\int_0^1 \frac{\rho'(x)S_p}{\rho(x)S_p'}\left(S_p^p - \frac{1}{p}\right)dx = o(1), \text{ as } n\to\infty.$$

Hence

$$\theta(1,\lambda_n) = c_\rho(1) + \frac{(p-1)}{p^2\lambda_n^{1/p}}\int_0^1 \frac{\rho'(x)S_p}{\rho(x)S_p'}dx + O\left(\frac{1}{\lambda_n^{1-\frac{2}{p}}}\right).$$

Then, we have

$$\theta(1,\lambda_n) = \frac{n\pi_p}{\lambda_n^{1/p}} = c_\rho(1) + \frac{(p-1)}{p^2\lambda_n^{1/p}}\int_0^1 \frac{\rho'(x)S_p}{\rho(x)S_p'}dx + O\left(\frac{1}{\lambda_n^{1-\frac{2}{p}}}\right),$$

or

$$\frac{1}{\lambda_n^{1/p}} = \frac{c_\rho(1)}{n\pi_p} + \frac{(p-1)}{p^2 n\pi_p \lambda_n^{1/p}} \int_0^1 \frac{\rho'(x)S_p}{\rho(x)S_p'}dx + \frac{1}{n\pi_p} O\left(\frac{1}{\lambda_n^{1-\frac{2}{p}}}\right),$$

from the boundary condition (1.2). As n is sufficiently large, it follows that

$$\frac{1}{\lambda_n^{1/p}} = \frac{c_\rho(1)}{n\pi_p} + \frac{(p-1)c_\rho(1)}{p^2 (n\pi_p)^2} \int_0^1 \frac{\rho'(x)S_p}{\rho(x)S_p'}dx + O\left(\frac{1}{n^p}\right). \tag{2.5}$$

Therefore, we get

$$\lambda_n^{1/p} = \frac{n\pi_p}{c_\rho(1)} + \frac{(p-1)}{p^2 c_\rho(1)} \int_0^1 \frac{\rho'(x)S_p}{\rho(x)S_p'}dx + O\left(\frac{1}{n^{p-2}}\right).$$

So, it completes the proof. □

Theorem 2.2. *Asymptotic estimate of the nodals for the problem* (1.1), (1.2) *satisfies*

$$\begin{aligned} x_j^n = \; & \frac{jc_\rho(1)}{n} + \frac{j(p-1)c_\rho(1)}{p^2 n^2 \pi_p} \int_0^1 \frac{\rho'(x)S_p}{\rho(x)S_p'}dx - \frac{c_\rho(1)}{p\,(n\pi_p)} \int_0^{x_j^n} \frac{\rho'(x)S_p}{\rho(x)S_p'}dx \\ & + \frac{c_\rho(1)}{p\,(n\pi_p)} \int_0^{x_j^n} \frac{\rho'(x)S_p}{\rho(x)S_p'} S_p^p dx - \int_0^{x_j^n} \left[\rho^{1/p}(x) - 1\right] dx + O\left(\frac{j}{n^p}\right), \end{aligned} \tag{2.6}$$

as $n \to \infty$.

Proof. Integrating (2.3) from 0 to x_j^n yields

$$\frac{j\pi_p}{\lambda_n^{1/p}} = x_j^n + \int_0^{x_j^n} \left[\rho^{1/p}(x) - 1\right] dx + \frac{1}{p\lambda_n^{1/p}} \int_0^{x_j^n} \frac{\rho'(x)S_p}{\rho(x)S_p'}dx - \frac{1}{p\lambda_n^{1/p}} \int_0^{x_j^n} \frac{\rho'(x)S_p}{\rho(x)S_p'} S_p^p dx.$$

By considering (2.5), we obtain

$$\begin{aligned} x_j^n = \; & \frac{jc_\rho(1)}{n} + \frac{j(p-1)c_\rho(1)}{p^2 n^2 \pi_p} \int_0^1 \frac{\rho'(x)S_p}{\rho(x)S_p'}dx - \frac{c_\rho(1)}{p\,(n\pi_p)} \int_0^{x_j^n} \frac{\rho'(x)S_p}{\rho(x)S_p'}dx \\ & + \frac{c_\rho(1)}{p\,(n\pi_p)} \int_0^{x_j^n} \frac{\rho'(x)S_p}{\rho(x)S_p'} S_p^p dx - \int_0^{x_j^n} \left[\rho^{1/p}(x) - 1\right] dx + O\left(\frac{j}{n^p}\right). \end{aligned}$$

□

Theorem 2.3. *Asymptotic estimate of the nodal lengths for* (1.1), (1.2) *has the below relation*

$$
\begin{aligned}
(2.7)\quad l_j^n &= \frac{c_\rho(1)}{n}+\frac{(p-1)c_\rho(1)}{p^2n^2\pi_p}\int\limits_0^1\frac{\rho'(x)S_p}{\rho(x)S_p'}dx-\int\limits_{x_j^n}^{x_{j+1}^n}\left[\rho^{1/p}(x)-1\right]dx\\
&\quad-\frac{c_\rho(1)}{p\,(n\pi_p)}\int\limits_{x_j^n}^{x_{j+1}^n}\frac{\rho'(x)S_p}{\rho(x)S_p'}dx+\frac{c_\rho(1)}{p\,(n\pi_p)}\int\limits_{x_j^n}^{x_{j+1}^n}\frac{\rho'(x)S_p}{\rho(x)S_p'}S_p^p dx+O\left(\frac{1}{n^p}\right),
\end{aligned}
$$

as $n\to\infty$.

Proof. For large $n\in\mathbb{N}$, integrating (2.3) in $[x_j^n,x_{j+1}^n]$ and using the definition for nodal length, we have

$$
\begin{aligned}
\frac{\pi_p}{\lambda_n^{1/p}} &= x_{j+1}^n-x_j^n+\int\limits_{x_j^n}^{x_{j+1}^n}\left[\rho^{1/p}(x)-1\right]dx+\frac{1}{p\lambda_n^{1/p}}\int\limits_{x_j^n}^{x_{j+1}^n}\frac{\rho'(x)S_p}{\rho(x)S_p'}dx\\
&\quad-\frac{1}{p\lambda_n^{1/p}}\int\limits_{x_j^n}^{x_{j+1}^n}\frac{\rho'(x)S_p}{\rho(x)S_p'}S_p^p dx,
\end{aligned}
$$

or

$$
\begin{aligned}
l_j^n &= \frac{c_\rho(1)}{n}+\frac{(p-1)c_\rho(1)}{p^2n^2\pi_p}\int\limits_0^1\frac{\rho'(x)S_p}{\rho(x)S_p'}dx-\int\limits_{x_j^n}^{x_{j+1}^n}\left[\rho^{1/p}(x)-1\right]dx\\
&\quad-\frac{c_\rho(1)}{p\,(n\pi_p)}\int\limits_{x_j^n}^{x_{j+1}^n}\frac{\rho'(x)S_p}{\rho(x)S_p'}dx+\frac{c_\rho(1)}{p\,(n\pi_p)}\int\limits_{x_j^n}^{x_{j+1}^n}\frac{\rho'(x)S_p}{\rho(x)S_p'}S_p^p dx+O\left(\frac{1}{n^p}\right).
\end{aligned}
$$

□

3. Reconstruction formula for the density function. In this section, we express an explicit formula for density function of p-Laplacian string equation by nodal parameters. The way that we used in the proof of the next theorem is similar to classical; p-Laplacian SL and energy-dependent SL eigenvalue problems (see [2]).

Theorem 3.1. *Assume that* ρ *is a positive* C^2*-function on* $[0,1]$. *Then*

$$
\rho^{1/p}(x)=\lim_{n\to\infty}\left[1+\frac{(1-p)}{p^2\pi_p}\int\limits_{x_j^n}^{x_{j+1}^n}\frac{\rho'(t)S_p}{\rho(t)S_p'}dt\right],
$$

for $j=j_n(x)=\max\left\{j:x_j^n<x\right\}$.

Proof. We use Theorem 2.3 to derive reconstruction formula of density function. After some calculations, we get

$$\begin{aligned} l_j^n &= \frac{\pi_p}{\lambda_n^{1/p}} - \int\limits_{x_j^n}^{x_{j+1}^n} \left[\rho^{1/p}(t) - 1\right] dt - \frac{1}{p\lambda_n^{1/p}} \int\limits_{x_j^n}^{x_{j+1}^n} \frac{\rho'(t) S_p}{\rho(t) S_p'} dt \\ &+ \frac{1}{p^2 \lambda_n^{1/p}} \int\limits_{x_j^n}^{x_{j+1}^n} \frac{\rho'(t) S_p}{\rho(t) S_p'} dt + \frac{1}{p\lambda_n^{1/p}} \int\limits_{x_j^n}^{x_{j+1}^n} \frac{\rho'(t) S_p}{\rho(t) S_p'} \left(S_p^p - \frac{1}{p} \right) dt. \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{\lambda_n^{1/p}}{\pi_p} l_j^n &= 1 - \frac{\lambda_n^{1/p}}{\pi_p} \int\limits_{x_j^n}^{x_{j+1}^n} \rho^{1/p}(t) dt + \frac{\lambda_n^{1/p}}{\pi_p} l_j^n + \frac{(1-p)}{p^2 \pi_p} \int\limits_{x_j^n}^{x_{j+1}^n} \frac{\rho'(t) S_p}{\rho(t) S_p'} dt \\ &+ \frac{1}{p\pi_p} \int\limits_{x_j^n}^{x_{j+1}^n} \frac{\rho'(t) S_p}{\rho(t) S_p'} \left(S_p^p - \frac{1}{p} \right) dt. \end{aligned}$$

Then, we can use similar technique as those in [9] for $j = j_n(x) = \max\left\{j : x_j^n < x\right\}$ to indicate

$$\frac{\lambda_n^{1/p}}{\pi_p} \int\limits_{x_j^n}^{x_{j+1}^n} \rho^{1/p}(t) dt \to \rho^{1/p}(x),$$

and

$$\frac{1}{p\pi_p} \int\limits_{x_j^n}^{x_{j+1}^n} \frac{\rho'(t) S_p}{\rho(t) S_p'} \left(S_p^p - \frac{1}{p} \right) dt \to 0,$$

pointwise convergent almost everywhere. Hence, we get

$$\rho^{1/p}(x) = \lim_{n\to\infty} \left[1 + \frac{(1-p)}{p^2 \pi_p} \int\limits_{x_j^n}^{x_{j+1}^n} \frac{\rho'(t) S_p}{\rho(t) S_p'} dt \right].$$

□

4. Conclusion. String equation has a very important place in physics. In the meantime, roughly speaking, string theory replaces point particles by strings, which can be either open or closed. In the classical sense, many results have been obtained regarding this equation. In this study, the classical equation is written in

Laplacian form and the inverse problem is solved by using Prüfer transform under some boundary conditions. Obtained results generalize the classical situation. These results can be used to solve some important problems in spectral theory.

REFERENCES

[1] Binding P., P. Drábek (2003) Sturm–Liouville theory for the p-Laplacian, Stud. Sci. Math. Hung., **40**, 375–396.

[2] Brown B. M., W. Reichel (2004) Eigenvalues of the radially symmetric p-Laplacian in $\mathbb{R}^n$, J. London Math. Soc., **59**, 657–675.

[3] Chen H. Y. (2009) On generalized trigonometric functions, MS Thesis, National Sun Yat-sen University, Kaohsiung, Taiwan.

[4] Dabbaghian A. H. (2018) Inverse nodal problem for p-Laplacian with two potential functions, Comput. Methods Differ. Equ., **6**(1), 19–29.

[5] Elbert A. (1987) On the half-linear second order differential equations, Acta Math. Hungarica, **49**, 487–508.

[6] Gulsen T., E. Yilmaz, H. Koyunbakan (2017) Inverse nodal problem for p-Laplacian Dirac system, Math. Methods Appl. Sci., **40**(7), 2329–2335.

[7] Hald O. H., J. R. McLaughlin (1989) Solution of the inverse nodal problems, Inverse Probl., **5**, 307–347.

[8] Koyunbakan H., E. Yilmaz (2008) Reconstruction of the potential function and its derivatives for the diffusion operator, Z. Naturforsch. A, **63**, 127–130.

[9] Law C. K., W. C. Lian, W. C. Wang (2009) The inverse nodal problem and the Ambarzumyan problem for the p-Laplacian, Proc. R. Soc. Edinb. A: Math., **139**(6), 1261–1273.

[10] Law C. K., C. F. Yang (1999) Reconstruction of the potential function and its derivatives using nodal data, Inverse Probl., **14**, 299–312.

[11] McLaughlin J. R. (1988) Inverse spectral theory using nodal points as data – a uniqueness result, J. Differ. Equ., **73**, 354–362.

[12] Pinasco J. P. (2006) Comparison of eigenvalues for the p-Laplacian with integral inequalities, Appl. Math. Comput., **182**(2), 1399–1404.

[13] Shen C. L. (1988) On the nodal sets of the eigenfunctions of the string equation, SIAM J. Math. Anal., **19**, 1419–1424.

[14] Wang W. C. (2010) Direct and inverse problems for one dimensional p-Laplacian operators, PhD Thesis, National Sun Yat-sen University, Kaohsiung, Taiwan.

[15] Wang W. C., Cheng Y. H., W. C. Lian (2011) Inverse nodal problems for the p-Laplacian with eigenparameter dependent boundary conditions, Math. Comput. Model., **54**(11–12), 2718–2724.

[16] Yang C. F. (2010) Reconstruction of the diffusion operator from nodal data, Z. Naturforsch. A, **65**(1), 100–106.

[17] Yantir A. (2004) Oscillation theory for second order differential equations and dynamic equations on time scales, MS Thesis, Izmir institue of Technology, Izmir.

[18] Yilmaz E., H. Koyunbakan (2014) On the high order Lipschitz stability of inverse nodal problem for string equation, Dyn. Contin., Discrete Impuls. Syst. A: Math. Anal., **21**, 79–88.

[19] Yurko V. A. (2008) Inverse nodal problems for Sturm–Liouville operators on star-type graphs, J. Inverse Ill-Posed Probl., **16**, 715–722.

Firat University
Department of Mathematics
23119 Elazıg, Turkey
e-mail: emrah231983@gmail.com
tubagulsen87@hotmail.com
hkoyunbakan@gmail.com