

## ON PACKING OF MINKOWSKI BALLS

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### Abstract

We investigate lattice packings of Minkowski balls. By the results of the proof of Minkowski conjecture about the critical determinant we divide Minkowski balls into 3 classes: Minkowski balls, Davis balls and Chebyshev–Cohn balls. We investigate lattice packings of these balls on planes with varying Minkowski metric and search among these packings the optimal packings. In this paper we prove that the optimal lattice packing of the Minkowski, Davis, and Chebyshev–Cohn balls is realized with respect to the sublattices of index two of the critical lattices of corresponding balls.

**Key words:** lattice packing, Minkowski ball, Minkowski metric, critical lattice, optimal lattice packing

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**1. Introduction.** A system of equal balls in  $n$ -dimensional space is said to form a packing, if no two balls of the system have any inner points in common.

Lately the remarkable results in resolving the problem of optimal packing of balls in 8 and 24-dimensional real Euclidean spaces have been obtained [1,2].

In this research, acknowledged by the Fields medal, optimal packings are constructed on lattices.

Our considerations connect with Minkowski conjecture [3–12] and use results of its proof. Corresponding results and conjectures are most simply stated in terms of geometric lattices and critical lattices [3,13–15]. The last are important partial case of geometric lattices. We investigate lattice packings of Minkowski balls,

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Davis balls and Chebyshev–Cohn balls on planes with varying Minkowski metric and search among these packings the optimal packings. The packing problem is studied on classes of lattices related to the problem of the theory of Diophantine approximations considered by MINKOWSKI [3] for the case of the plane. For some other selected problems and results of the theory Diophantine approximations see for instance [16] and references therein. The naming of balls is connected with results of investigating of Minkowski conjecture on critical determinant and its justification is given below. In this paper we prove that the optimal lattice packing of the Minkowski, Davis, and Chebyshev–Cohn balls is realized with respect to the sublattices of index two of the critical lattices of corresponding balls.

## 2. Minkowski conjecture, its proof and Minkowski balls. Let

$$|\alpha x + \beta y|^p + |\gamma x + \delta y|^p \leq c \cdot |\det(\alpha\delta - \beta\gamma)|^{p/2},$$

be a diophantine inequality defined for a given real  $p > 1$ ; here  $\alpha, \beta, \gamma, \delta$  are real numbers with  $\alpha\delta - \beta\gamma \neq 0$ .

Minkowski in his monograph [3] raises the question about minimum constant  $c$  such that the inequality has integer solution other than origin. Minkowski with the help of his theorem on convex body has found a sufficient condition for the solvability of Diophantine inequalities in integers not both zero:

$$c = \kappa_p^p, \kappa_p = \frac{\Gamma(1 + \frac{2}{p})^{1/2}}{\Gamma(1 + \frac{1}{p})}.$$

But this result is not optimal, and Minkowski also raised the issue of not improving constant  $c$ . For this purpose Minkowski has proposed to use the critical determinant.

Recall the definitions [14].

Let  $\mathcal{R}$  be a set and  $\Lambda$  be a lattice with base

$$\{a_1, \dots, a_n\}$$

in  $\mathbb{R}^n$ .

A lattice  $\Lambda$  is *admissible* for body  $\mathcal{R}$  ( *$\mathcal{R}$ -admissible*) if  $\mathcal{R} \cap \Lambda = \emptyset$  or  $0$ .

Let  $\overline{\mathcal{R}}$  be the closure of  $\mathcal{R}$ . A lattice  $\Lambda$  is *strictly admissible* for  $\overline{\mathcal{R}}$  ( *$\overline{\mathcal{R}}$ -strictly admissible*) if  $\overline{\mathcal{R}} \cap \Lambda = \emptyset$  or  $0$ .

Let

$$d(\Lambda) = |\det(a_1, \dots, a_n)|$$

be the determinant of  $\Lambda$ .

The infimum  $\Delta(\mathcal{R})$  of determinants of all lattices admissible for  $\mathcal{R}$  is called *the critical determinant* of  $\mathcal{R}$ ; if there are no  $\mathcal{R}$ -admissible lattices, then put  $\Delta(\mathcal{R}) = \infty$ . A lattice  $\Lambda$  is *critical* if  $d(\Lambda) = \Delta(\mathcal{R})$ .

For the given 2-dimension region  $D_p \subset \mathbb{R}^2 = (x, y)$ ,  $p \geq 1$ :

$$|x|^p + |y|^p < 1,$$

let  $\Delta(D_p)$  be the critical determinant of the region.

All determinants of admissible lattices of this domain that have three pairs of points on the boundary of this domain are parametrized by the Minkowski–Cohn moduli space of the form

$$(1) \quad \Delta(p, \sigma) = (\tau + \sigma)(1 + \tau^p)^{-\frac{1}{p}}(1 + \sigma^p)^{-\frac{1}{p}},$$

in the domain

$$\mathcal{M} : \infty > p > 1, 1 \leq \sigma \leq \sigma_p = (2^p - 1)^{\frac{1}{p}},$$

of the  $\{p, \sigma\}$ -plane, where  $\sigma$  is some real parameter [6, 10–12, 17].

In notations [12] the following result has been proved:

**Theorem 1** ([12]).

$$\Delta(D_p) = \begin{cases} \Delta(p, 1), & 1 \leq p \leq 2, p \geq p_0, \\ \Delta(p, \sigma_p), & 2 \leq p \leq p_0; \end{cases}$$

Here  $p_0$  is a real number that is defined unique by conditions  $\Delta(p_0, \sigma_p) = \Delta(p_0, 1)$ ,  $2.57 < p_0 < 2.58$ ,  $p_0 \approx 2.5725$ .

**Remark 1.** We will call  $p_0$  the Davis constant.

**Corollary 1.**

$$\kappa_p = \Delta(D_p)^{-\frac{p}{2}}.$$

From Theorem (1) in notations [12, 17] we deduce the following corollary:

**Corollary 2** ([12, 17]).

$$\Delta_p^{(0)} = \Delta(p, \sigma_p) = \frac{1}{2}\sigma_p, \quad \sigma_p = (2^p - 1)^{1/p},$$

$$\Delta_p^{(1)} = \Delta(p, 1) = 4^{-\frac{1}{p}} \frac{1 + \tau_p}{1 - \tau_p}, \quad 2(1 - \tau_p)^p = 1 + \tau_p^p, \quad 0 \leq \tau_p < 1.$$

For their critical lattices, respectively,  $\Lambda_p^{(0)}$ ,  $\Lambda_p^{(1)}$  next conditions satisfy:  $\Lambda_p^{(0)}$  and  $\Lambda_p^{(1)}$  are two  $D_p$ -admissible lattices each of which contains three pairs of points on the boundary of  $D_p$  with the property that

$$(1, 0) \in \Lambda_p^{(0)}, \quad (-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)},$$

(under these conditions the lattices are uniquely defined).

**3. Minkowski balls and the density of the packing of 2-dimensional Minkowski balls.** Recall the Davis constant  $p_0 \in \mathbb{R}$ , satisfying  $2.57 < p_0 < 2.58$ . We consider balls of the form

$$D_p : |x|^p + |y|^p \leq 1, \quad p \geq 1,$$

and call such balls *Minkowski balls* and, correspondingly, the circles defined by

$$|x|^p + |y|^p = 1, \quad p \geq 1$$

are called *Minkowski circles*.

According to Theorem 1, the half-line  $p > 1$  is divided into segments according to the value of the determinant of the critical lattice. Following this division, which is determined by the value of the critical determinant of the lattice, we specify the name of the Minkowski balls, leaving the name of the Minkowski ball (with respect to its critical lattice) for the domain  $1 < p < 2$  and call such balls with  $1 < p < 2$  *Minkowski balls* and, correspondingly, the circles defined by

$$|x|^p + |y|^p = 1, \quad 1 < p < 2$$

are called *Minkowski circles*.

Continuing this, we consider the following classes of balls and circles.

- *Limit Minkowski circles:*  $|x| + |y| = 1$ ;
- *Davis balls:*  $|x|^p + |y|^p \leq 1$  for  $p_0 > p \geq 2$ ;
- *Davis circles:*  $|x|^p + |y|^p = 1$  for  $p_0 > p \geq 2$ ;
- *Chebyshev–Cohn balls:*  $|x|^p + |y|^p \leq 1$  for  $p \geq p_0$ ;
- *Chebyshev–Cohn circles:*  $|x|^p + |y|^p = 1$  for  $p \geq p_0$ ;
- *Limit Chebyshev–(Cohn) balls:*  $\|x, y\|_\infty = \max(|x|, |y|)$ .

Recall the definition of a packing lattice [1,2,14,15]. We will give it for  $n$ -dimensional Minkowski balls  $D_p^n$  in  $\mathbb{R}^n$ .

**Definition 1.** Let  $\Lambda$  be a full lattice in  $\mathbb{R}^n$  and  $a \in \mathbb{R}^n$ . In the case if it occurs that no two of balls  $\{D_p^n + b, b \in \Lambda + a\}$  have inner points in common, the collection of balls  $\{D_p^n + b, b \in \Lambda + a\}$  is called a  $(D_p^n, \Lambda)$ -packing, and  $\Lambda$  is called a packing lattice of  $D_p^n$ .

Recall also that if  $\alpha \in \mathbb{R}$  and  $D_p^n$  is a ball, then  $\alpha D_p^n$  is the set of points  $\alpha x, x \in D_p^n$ .

In some cases we will consider interiors of balls  $D_p = D_p^2$  (open balls) which we will denote as  $ID_p$ .

From the considerations of Minkowski and other authors [3, 13–15, 17], the following statements can be deduced (for the sake of completeness, we present the proof of Proposition 1).

Denote by  $V(D_p)$  the volume (area) of  $D_p$ .

**Proposition 1** ([14, 15]). *A lattice  $\Lambda$  is a packing lattice of  $D_p$  if and only if it is admissible lattice for  $2D_p$ .*

**Proof** (contrary proof). First, note that one can take an open ball  $ID_p$  and use the notion of strict admissibility. Suppose that  $\Lambda$  is not strictly admissible for  $2ID_p$ . Then  $2ID_p$  contains a point  $a \neq 0$  of  $\Lambda$ . Then the two balls  $ID_p$  and  $ID_p + a$  contain the point  $\frac{1}{2}a$  in common. So  $\Lambda$  is not a packing lattice of  $D_p$ .

Suppose now that  $\Lambda$  is not a packing lattice of  $D_p$ . Then there exist two distinct points  $b_1, b_2 \in \Lambda$  and a point  $c$  such that  $c \in ID_p + b_1$  and  $c \in ID_p + b_2$ . Hence there are points  $a_1, a_2 \in ID_p$  such that  $c = a_1 + b_1 = a_2 + b_2$ . So  $b_1 - b_2 = a_2 - a_1 \in ID_p$ , whereas  $b_1 - b_2 \neq 0$  and  $b_1 - b_2 \in \Lambda$ . Therefore  $\Lambda$  is not (strictly) admissible lattice.  $\square$

**Proposition 2** ([14, 15]). *The density of a  $(D_p, \Lambda)$ -packing is equal to  $V(D_p)/d(\Lambda)$  and it is maximal if  $\Lambda$  is critical for  $2D_p$ .*

**4. On packing Minkowski balls, Davis balls and Chebyshev–Cohn balls on the plane.** Let us consider possible optimal lattice packings of these balls and their connection with critical lattices.

At first give lattices of trivial optimal lattice packings for the limiting (asymptotic) cases at the points  $p = 1$  and  $\infty$  “infinity” (the latter corresponds to the classical Chebyshev balls) and as the introductory example the optimal packing of two-dimensional unit balls.

**Proposition 3.** *The lattice*

$$\Lambda_1^{(1)} = \left\{ \left( \frac{1}{2}, \frac{1}{2} \right), (0, 1) \right\}$$

*is the critical lattice for  $D_1$ . Limiting case of Minkowski balls for  $p = 1$  gives the optimal sublattice of index two of the lattice  $\Lambda_1^{(1)}$ -lattice packing with the density 1. The centres of the Minkowski balls in this case are at the vertices of the sublattice of index two of the lattice  $\Lambda_1^{(1)}$ .*

**Proof.** Recall that a critical lattice for  $D_1$  is a lattice  $\Lambda$  which is  $D_1$ -admissible and which has determinant  $d(\Lambda) = \Delta(D_1)$ . The lattice  $\Lambda_1^{(1)}$  is  $D_1$ -admissible. We have  $\Delta(D_1) = \frac{1}{2}$  and  $d(\Lambda_1^{(1)}) = \frac{1}{2}$ . Minkowski balls for  $p = 1$  are congruent squares. Hence we have the optimal sublattice of index two of the lattice  $\Lambda_1^{(1)}$  packing of the squares with the density 1.  $\square$

**Proposition 4.** *The lattice*

$$\Lambda_\infty^{(1)} = \{(1, 1), (0, 1)\}$$

is the critical lattice for  $D_\infty$ . Limiting case of Minkowski balls for  $p = \infty$  gives the optimal of the density 1 packing with respect to the sublattice of index two of the critical lattice  $\Lambda_\infty^{(1)}$ . The centres of the Minkowski balls in this case are at the vertices of the sublattice of index two of the lattice  $\Lambda_\infty^{(1)}$ .

**Proof.** The lattice  $\Lambda_\infty^{(1)}$  is  $D_\infty$ -admissible. We have  $\Delta(D_\infty) = 1$  and  $d(\Lambda_\infty^{(1)}) = 1$ . Minkowski balls for  $p = \infty$  are congruent squares. Hence we have the optimal sublattice of index two of the lattice  $\Lambda_\infty^{(1)}$  packing of the squares with the density 1.  $\square$

**Proposition 5.** *The lattice*

$$\Lambda_2^{(0)} = \left\{ (1, 0), \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right\}$$

is the critical lattice for  $D_2$ . The lattice packing of Davis balls for  $p = 2$  gives the optimal of the density  $\approx 0.91$  packing with respect to the sublattice of index two of the critical lattice  $\Lambda_2^{(0)}$ . The centres of the Minkowski balls in this case are at the vertices of the sublattice of index two of the lattice  $\Lambda_2^{(0)}$ .

**Proof.** As in Proposition 3 a critical lattice for  $D_2$  is a lattice  $\Lambda$  which is  $D_2$ -admissible and which has determinant  $d(\Lambda) = \Delta(D_2)$ . The lattice  $\Lambda_2^{(0)}$  is  $D_2$ -admissible. We have  $\Delta(D_2) = \frac{\sqrt{3}}{2}$  and  $d(\Lambda_2^{(0)}) = \frac{\sqrt{3}}{2}$ . So sublattice of index two of the lattice  $\Lambda_2^{(0)}$  is the hexagonal lattice. Next, we use the following classical results [18, 19]: the optimal sphere packing of dimension 2 is the hexagonal lattice (honeycomb) packing with the density  $\approx 0.91$ .  $\square$

**Proposition 6.** *If  $\Lambda$  is the critical lattice of the Minkowski ball  $D_p$ , then the sublattice  $\Lambda_2$  of index two of the critical lattice is the critical lattice of  $2D_p$ . (Examples for  $n = 1, 2, \infty$  above).*

Here we give the proof of Proposition 6.

**Proof.** Since the Minkowski ball  $D_p$  is symmetric about the origin and convex, then  $2D_p$  is convex and symmetric about the origin [14, 15].

When parametrizing admissible lattices  $\Lambda$  having three pairs of points on the boundary of the ball  $D_p$ , the following parametrization is used [6, 11, 12, 17]:

$$(2) \quad \Lambda = \left\{ \left( (1 + \tau^p)^{-\frac{1}{p}}, \tau(1 + \tau^p)^{-\frac{1}{p}} \right), \left( -(1 + \sigma^p)^{-\frac{1}{p}}, \sigma(1 + \sigma^p)^{-\frac{1}{p}} \right) \right\},$$

where

$$0 \leq \tau < \sigma, \quad 0 \leq \tau \leq \tau_p.$$

$\tau_p$  is defined by the equation  $2(1 - \tau_p)^p = 1 + \tau_p^p$ ,  $0 \leq \tau_p < 1$ .

$$1 \leq \sigma \leq \sigma_p, \quad \sigma_p = (2^p - 1)^{\frac{1}{p}}.$$

Admissible lattices of the form (2) for doubled Minkowski balls  $2D_p$  have a representation of the form

$$(3) \quad \Lambda_{2D_p} = \{2((1 + \tau^p)^{-\frac{1}{p}}, 2\tau(1 + \tau^p)^{-\frac{1}{p}}), (-2(1 + \sigma^p)^{-\frac{1}{p}}, 2\sigma(1 + \sigma^p)^{-\frac{1}{p}})\}.$$

Hence the Minkowski–Cohn moduli space for these admissible lattices has the form

$$(4) \quad \Delta(p, \sigma)_{2D_p} = 4(\tau + \sigma)(1 + \tau^p)^{-\frac{1}{p}}(1 + \sigma^p)^{-\frac{1}{p}},$$

in the same domain

$$\mathcal{M} : \infty > p > 1, 1 \leq \sigma \leq \sigma_p = (2^p - 1)^{\frac{1}{p}}.$$

Consequently, the critical determinants of doubled Minkowski balls have a representation of the form

$$\Delta_p^{(0)}(2D_p) = \Delta(p, \sigma_p)_{2D_p} = 2 \cdot \sigma_p, \sigma_p = (2^p - 1)^{1/p},$$

$$\Delta_p^{(1)}(2D_p) = \Delta(p, 1)_{2D_p} = 4^{1-\frac{1}{p}} \frac{1 + \tau_p}{1 - \tau_p}, 2(1 - \tau_p)^p = 1 + \tau_p^p, 0 \leq \tau_p < 1.$$

And these are the determinants of the sublattices of index 2 of the critical lattices of the corresponding Minkowski balls.  $\square$

**Theorem 2.** *The optimal lattice packing of the Minkowski, Davis, and Chebyshev–Cohn balls is realized with respect to the sublattices of index two of the critical lattices*

$$(1, 0) \in \Lambda_p^{(0)}, (-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)}.$$

**Proof.** By Proposition 6 the critical lattice of  $2D_p$  is the sublattice of index two of the critical lattice of Minkowski ball  $D_p$ .

So it is the admissible lattice for  $2D_p$  and by Proposition 1 is packing lattice of  $D_p$ . By Proposition 2 the corresponding lattice packing has maximal density and so is optimal.  $\square$

**Remark 2.** This result concerns the packing of unit balls and spheres in complete normed (Banach) spaces of dimension 2.

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## REFERENCES

- [<sup>1</sup>] VIAZOVSKA M. S. (2017) The sphere packing problem in dimension 8, *Ann. of Math.*, (2), **185**(3), 991–1015.

- [2] COHN H., A. KUMAR, S. D. MILLER, D. RADCHENKO, M. VIAZOVSKA (2017) The sphere packing problem in dimension 24, *Ann. of Math. (2)*, **185**(3), 1017–1033.
- [3] MINKOWSKI H. (1907) *Diophantische Approximationen. Eine Einführung in die Zahlentheorie*, Leipzig, B. G. Teubner.
- [4] MORDELL L. J. (1941) Lattice points in the region  $|Ax^4| + |By^4| \geq 1$ , *J. London Math. Soc.*, **16**, 152–156.
- [5] DAVIS C. (1948) Note on a conjecture by Minkowski, *J. London Math. Soc.*, **23**, 172–175.
- [6] COHN H. (1950) Minkowski's conjectures on critical lattices in the metric  $\{|\xi|^p + |\eta|^p\}^{1/p}$ , *Ann. of Math. (2)*, **51**(2), 734–738.
- [7] WATSON G. (1953) Minkowski's conjecture on the critical lattices of the region  $|x|^p + |y|^p \leq 1$  (I), (II), *J. London Math. Soc.*, **28**, 305–309, 402–410.
- [8] MALYSHEV A. (1977) The application of an electronic computer to the proof of a certain conjecture of Minkowski from the geometry of numbers, *Zap. Nauch. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)*, **71**, 163–180 (in Russian).
- [9] MALYSHEV A. (1979) The application of an electronic computer to the proof of a certain conjecture of Minkowski from the geometry of numbers, II, *Zap. Nauch. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)*, **82**, 29–32 (in Russian).
- [10] GLAZUNOV N., A. V. MALYSHEV (1985) On Minkowski's critical determinant conjecture, *Kibernetika (Kiev)*, no. 5, 10–14, (in Russian).
- [11] GLAZUNOV N., A. MALYSHEV (1986) A new proof of the Minkowski conjecture on the critical determinant of the region  $|x|^p + |y|^p < 1$  in a neighborhood of  $p = 2$ , *Dokl. Akad. Nauk Ukrain. SSR Ser. A*, no. 7, 9–12, (in Russian).
- [12] GLAZUNOV N., A. GOLOVANOV, A. MALYSHEV (1986) Proof of Minkowski's hypothesis about the critical determinant of  $|x|^p + |y|^p < 1$  domain, *Research in Number Theory*, 9, Notes of scientific seminars of LOMI, **151**, Leningrad, Nauka, 40–53.
- [13] MINKOWSKI H. (1910) *Geometrie der Zahlen*, Berlin–Leipzig, B. G. Teubner.
- [14] CASSELS J. W. S. (1997) *An Introduction to the Geometry of Numbers*, Corrected reprint of the 1971 edition, *Classics in Mathematics*, Berlin, Springer-Verlag.
- [15] LEKKERKERKER C. G. (1969) *Geometry of Numbers*, *Bibliotheca Mathematica*, Vol. VIII, Groningen: Wolters-Noordhoff Publishing; Amsterdam-London: North-Holland Publishing Company.
- [16] ANDERSEN N., W. DUKE (2021) On a theorem of Davenport and Schmidt, *Acta Arith.*, **198**(1), 37–75.
- [17] GLAZUNOV N. (2016) On A. V. Malyshev's approach to Minkowski's conjecture concerning the critical determinant of the region  $|x|^p + |y|^p < 1$  for  $p > 1$ , *Chebyshevskii Sb.*, **17**(4), 185–193.
- [18] FEJES L. (1940) Über einen geometrischen Satz, *Math. Z.*, **46**, 83–85. <https://link.springer.com/article/10.1007/BF01181430>
- [19] THUE A. (1910) Über die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene. *Christiania Vid.-Selsk. Skr.* 1910, Nr. 1, 9 s.

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