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ANOTHER CLASS OF CR-SLANT WARPED PRODUCT SUBMANIFOLDS IN NEARLY TRANS-SASAKIAN MANIFOLDS

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Abstract

In this paper, we study a CR-slant warped product submanifold of the type $B \times_f M_\theta$, where M_θ is a slant and B is a CR-product submanifold in a nearly trans-Sasakian manifold. We obtain an inequality for the squared norm of the second fundamental form in two cases, depending on the structure vector field affiliation, in such warped products. The equality case is also considered.

Key words: warped products, CR-warped product, CR-slant warped product, slant submanifolds, nearly Sasakian, nearly Kenmotsu, nearly trans-Sasakian manifolds

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1. Introduction Sahin introduced the notion of CR-slant warped products under the name of skew CR-warped products in Kaehler manifolds in [1]. After that, Chen, Uddin and Al-Solamy worked on the pointwise CR-slant warped product of Kaehler manifolds in [2]. Also, the CR-slant warped product was studied in nearly cosymplectic manifolds [3].

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In nearly trans-Sasakian manifold, two types of CR-slant warped product were considered in [4]. Here, we study a new type of a CR-slant warped product of the form $B \times_f M_\theta$, where B is a CR-product submanifold and M_θ is a slant submanifold of nearly trans-Sasakian manifolds. We obtain a geometric inequality which relates the second fundamental form and the warping function in two different cases depending on the structure vector field affiliation. Also, the case where the equality holds is discussed.

2. Preliminaries Let $\bar{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with almost contact structure (ϕ, ξ, η), where ϕ is a $(1,1)$ tensor field, ξ is a vector field and η is a 1-form satisfying the following properties [5]

$$\phi(\xi) = 0, \quad \phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta(\phi) = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \text{and} \quad \eta(X) = g(X,\xi) \tag{2.2}$$

for any $X, Y \in \Gamma(T\bar{M})$. The structure (ϕ, ξ, η, g) is called an almost contact metric structure. An almost contact metric manifold $\bar{M}$ with almost contact metric structure (ϕ, ξ, η, g) is called nearly trans-Sasakian manifold if [6]

$$\begin{aligned}\left(\bar{\nabla}_X \phi\right) Y + \left(\bar{\nabla}_Y \phi\right) X = \alpha \left(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\right) \\ - \beta \left(\eta(Y)\phi X + \eta(X)\phi Y\right)\end{aligned} \tag{2.3}$$

for some functions α and β on $\bar{M}$ and so, the trans-Sasakian manifold $\bar{M}$ is called of type (α, β). On $\bar{M}$, the covariant derivative of ϕ is defined as

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y, \tag{2.4}$$

for any $X, Y \in \Gamma(T\bar{M})$. Let M be an n-dimensional submanifold of $\bar{M}$. Then, for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, the Gauss–Weingarten formulas are given by [7]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \bar{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.5}$$

respectively, where h is the second fundamental form and A_N is the shape operator of M which are related as [7]

$$g(h(X,Y), N) = g(A_N X, Y), \tag{2.6}$$

where the induced metric on M from g is also denoted by g. The gradient of a smooth function f is denoted by $\operatorname{grad} f$ and is defined by [7]

$$g(\operatorname{grad} f, X) = X(f) \quad \text{and so} \quad \|\operatorname{grad} f\|^2 = \sum_{i=1}^{n} (E_i(f))^2 \tag{2.7}$$

for any $X \in \Gamma(TM)$, where $\{E_1, \ldots, E_n\}$ is an orthonormal frame on M. For any point $p \in M$, let $\{E_1, \ldots, E_n, \ldots, E_{2m+1}\}$ be an orthonormal frame for the tangent

space $T_p\bar{M}$, where $E_1, \ldots, E_n$ are tangent to M at p. The second fundamental form h is defined as follows [5]

$$\|h\|^2 = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} g(h(E_i, E_j), E_r)^2. \tag{2.8}$$

The submanifold M is said to be totally geodesic if $h(X, Y) = 0$ and totally umbilical if $h(X, Y) = g(X, Y)H$, where H is the mean curvature on M [8]. For any $X \in \Gamma(TM)$, the vector field ϕX decomposes into tangential and normal components, respectively, as

$$\phi X = PX + FX. \tag{2.9}$$

The submanifold M is called invariant (anti-invariant) if

$$\phi(T_pM) \subseteq T_pM \quad (\phi(T_pM) \subseteq T_p^{\perp}M)$$

for each $p \in M$. Let D_T and $D_{\perp}$ denote the invariant and anti-invariant distributions on M, respectively. If $TM = D_T \oplus D_{\perp} \oplus \langle \xi \rangle$, then M is called a CR-submanifold [9]. Now, let $X \in T_pM$ such that it is not proportional to ξ. Then, the angle between ϕX and T_pM is called a slant angle and is denoted by θ. If θ is constant, then M is called a slant submanifold [10]. Moreover, in [11] it is shown that M is slant if and only if there exists a constant $\lambda \in [0, 1]$, which satisfies

$$P^2 = \lambda(-Id + \eta \otimes \xi). \tag{2.10}$$

Furthermore, θ is the slant angle and satisfies $\lambda = \cos^2 \theta$. Also, the following relations are consequences of (2.10), for any $X, Y \in \Gamma(TM)$,

$$g(PX, PY) = \cos^2 \theta \left(g(X, Y) - \eta(X)\eta(Y)\right), \tag{2.11}$$

$$g(FX, FY) = \sin^2 \theta \left(g(X, Y) - \eta(X)\eta(Y)\right). \tag{2.12}$$

Now, let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f be a positive smooth function on M_1. Then the warped product manifold $M = M_1 \times_f M_2$ is a manifold with Riemannian metric $g = g_1 + f^2 g_2$. The function f is called the warping function on M [12]. For any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, we have

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z. \tag{2.13}$$

Furthermore, on warped product manifold $M = M_1 \times_f M_2$, M_1 is totally geodesic in M and M_2 is totally umbilical in M [5].

3. Basic results. Let M be a submanifold tangent to ξ of an almost contact metric manifold $\bar{M}$. Let M_T, $M_\perp$ and M_θ be invariant, anti-invariant and slant submanifolds of $\bar{M}$, respectively. M is said to be a CR-slant warped product submanifold if M is a warped product of one of the form $B_1 \times_{f_1} M_T$, where $B_1 = M_\perp \times M_\theta$, $B_2 \times_{f_2} M_\perp$, where $B_2 = M_T \times M_\theta$, or $B \times_f M_\theta$, where $B = M_T \times M_\theta$.

In nearly trans-Sasakian manifold, the first two forms of CR-slant warped products were studied in [4]. Here, we study the other type, where $M = B \times_f M_\theta$.

The tangent bundle of the CR-slant warped product submanifold is decomposed as

$$TM = D_T \oplus D_\perp \oplus D_\theta \oplus \langle \xi \rangle. \tag{3.1}$$

For the CR-slant warped product $M = B \times_f M_\theta$ of a nearly trans-Sasakian manifold $\bar{M}$, we have two cases: the first is when ξ is tangent to B and the second is when ξ is tangent to M_θ. The following theorem deals with the second case.

Theorem 3.1. *There does not exist any proper warped product submanifold $M = B \times_f M_\theta$ of a nearly trans-Sasakian manifold $\bar{M}$ of type (α, β) if ξ is tangent to the proper slant submanifold M_θ of $\bar{M}$, where B is any Riemannian submanifold of $\bar{M}$, unless $\bar{M}$ is nearly α-Sasakian.*

Proof. Let $X \in \Gamma(TB)$. Since $\xi \in \Gamma(TM_\theta)$, then by (2.3), we have

$$(\bar{\nabla}_X \phi)\xi + (\bar{\nabla}_\xi \phi)X = -\alpha X - \beta \phi X. \tag{3.2}$$

Using (2.4), (2.2), (2.5) and (2.13) in the left hand side of (3.2), we obtain

$$-2\phi h(X, \xi) + \bar{\nabla}_\xi \phi X = -\alpha X - \beta \phi X. \tag{3.3}$$

Taking the Riemannian metric g with ϕX in (3.3), we get

$$g(\bar{\nabla}_\xi \phi X, \phi X) = -\beta g(\phi X, \phi X). \tag{3.4}$$

By applying metric compatible property, (2.3), (2.5) and (2.13) in the left hand side of (3.4), we obtain $g(\bar{\nabla}_\xi \phi X, \phi X) = 0$, and so, $\beta \|X\|^2 = 0$. Hence, $\bar{M}$ is nearly α-Sasakian manifold. □

If we consider ξ tangent to B, then we have the following lemmas.

Lemma 3.1. *Let $\bar{M}$ be a nearly trans-Sasakian manifold of type (α, β) and $M = B \times_f M_\theta$ be a CR-slant warped product submanifold of $\bar{M}$, where $B = M_T \times M_\perp$ is a CR-product submanifold tangent to ξ and M_θ is a proper slant submanifold of $\bar{M}$. Then, for any $X \in \Gamma(TM_T)$, $Z, W \in \Gamma(TM_\perp)$ and $U \in \Gamma(TM_\theta)$, we have*

$$\xi \ln f = \beta, \tag{3.5}$$

$$g(h(X, Z), FU) = -g(h(X, U), \phi Z), \tag{3.6}$$

$$g(h(Z, W), FU) = g(h(U, W), \phi Z). \tag{3.7}$$

Proof. Let $U \in \Gamma(TM_\theta)$, then, by (2.3) and (2.4), we have

$$\bar{\nabla}_U \phi\xi - \phi\bar{\nabla}_U \xi + \bar{\nabla}_\xi \phi U - \phi\bar{\nabla}_\xi U = -\alpha U - \beta\phi U. \tag{3.8}$$

Using (2.5) and (2.13) in (3.8) and since $\phi\xi = 0$, we get

$$-\xi(\ln f)\phi U - 2\phi h(\xi, U) = -\alpha U - \beta\phi U. \tag{3.9}$$

Taking the Riemannian metric g with ϕU in (3.9), we get $\xi(\ln f) = \beta$, which proves (3.5). Now, let $Z \in \Gamma(TM_\perp)$, then by (2.3), we have

$$(\bar{\nabla}_Z \phi)U + (\bar{\nabla}_U \phi)Z = -\alpha\eta(Z)U - \beta\eta(Z)\phi U. \tag{3.10}$$

Applying (2.4), in the left hand side of (3.10) and using (2.9), (2.5) and (2.13), the equation (3.10) takes the form

$$\begin{aligned} &- Z(\ln f)PU + h(Z, PU) - A_{FU}Z + \nabla^\perp_Z FU - 2Z(\ln f)FU \\ &\quad - 2h(Z, U) - A_{\phi Z}U + \nabla^\perp_U \phi Z = -\alpha\eta(Z)U - \beta\eta(Z)PU - \beta\eta(Z)FU. \end{aligned} \tag{3.11}$$

Taking the Riemannian metric g with $X \in \Gamma(TM_T)$ in (3.11), we get (3.6). Similarly, by taking the Riemannian metric g with $W \in \Gamma(TM_\perp)$ in (3.11) and using (2.6), we get

$$g(h(Z, W), FU) + g(h(U, W), \phi Z) = 2g(h(Z, U), \phi W). \tag{3.12}$$

By using polarization identity in (3.12), we get

$$2g(h(Z, W), FU) - 4g(h(W, U), \phi Z) = -2g(h(U, Z), \phi W). \tag{3.13}$$

From (3.12) and (3.13), we prove (3.7). □

Lemma 3.2. *Let $\bar{M}$ be a nearly trans-Sasakian manifold of type (α, β) and $M = B \times_f M_\theta$ be a warped product CR-slant submanifold of $\bar{M}$, where $B = M_T \times M_\perp$ is a CR-product submanifold tangent to ξ and M_θ is a proper slant submanifold of $\bar{M}$. Then, for any $Z \in \Gamma(TM_\perp)$ and $U, V \in \Gamma(TM_\theta)$, we have*

$$\begin{aligned} g(h(U, V), \phi Z) &= g(h(Z, U), FV) + \alpha\eta(Z)g(U, V) \\ &\quad - \frac{1}{3}[Z(\ln f) - \beta\eta(Z)]g(PU, V). \end{aligned} \tag{3.14}$$

Proof. Let $Z \in \Gamma(TM_\perp)$ and $U, V \in \Gamma(TM_\theta)$. Taking the Riemannian metric g with $V \in \Gamma(TM_\theta)$ in (3.11) and using (2.6), we get

$$\begin{aligned} &- g(h(Z, V), FU) + 2g(h(Z, U), FV) - g(h(U, V), \phi Z) \\ &\quad = -\alpha\eta(Z)g(U, V) + [Z(\ln f) - \beta\eta(Z)]\, g(PU, V). \end{aligned} \tag{3.15}$$

By polarization identity of U, V in (3.15), we get

$$-g(h(Z,U),FV)+2g(h(Z,V),FU)-g(h(U,V),\phi Z) = -\alpha\eta(Z)g(U,V)+[Z(\ln f)-\beta\eta(Z)]\,g(PV,U). \tag{3.16}$$

From (3.15) and (3.16), we get (3.14), which proves the lemma. □

From Lemma 3.2, one can deduce the following equations: by interchanging V by PV (U by PU) in (3.14) and using (2.10) and (2.11), we get, respectively,

$$g(h(U,PV),\phi Z) = g(h(Z,U),FPV)+\alpha\eta(Z)g(U,PV) - \frac{1}{3}\cos^2\theta\,[Z(\ln f)-\beta\eta(Z)]\,g(U,V), \tag{3.17}$$

$$g(h(PU,V),\phi Z) = -g(h(Z,PU),FV)+\alpha\eta(Z)g(PU,V) + \frac{1}{3}\cos^2\theta\,[Z(\ln f)-\beta\eta(Z)]\,g(U,V). \tag{3.18}$$

Also, by interchanging V by PV in (3.18) and using (2.11), we get

$$g(h(PU,PV),\phi Z) = g(h(Z,PU),FPV)+\alpha\cos^2\theta\eta(Z)g(U,V) + \frac{1}{3}\cos^2\theta\,[Z(\ln f)-\beta\eta(Z)]\,g(U,PV). \tag{3.19}$$

From (3.17) and (3.18), one can see

$$g(h(U,PV),\phi Z)+g(h(PU,V),\phi Z) = g(h(Z,U),FPV)+g(h(Z,PU),FV). \tag{3.20}$$

Lemma 3.3. *Let $\bar{M}$ be a nearly trans-Sasakian manifold of type (α,β) and $M = B\times_f M_\theta$ be a warped product CR-slant submanifold of $\bar{M}$, where $B = M_T\times M_\perp$ is a CR-product submanifold tangent to ξ and M_θ is a proper slant submanifold of $\bar{M}$. Then, for any $X\in\Gamma(TM_T)$ and $U,V\in\Gamma(TM_\theta)$, we have*

$$g(h(U,X),FV) = (\phi X(\ln f)-\alpha\eta(X))\,g(U,V) - \frac{1}{3}\,(X(\ln f)-\beta\eta(X))\,g(PV,U). \tag{3.21}$$

Proof. Let $X\in\Gamma(TM_T)$ and $U,V\in\Gamma(TM_\theta)$. From (2.3), we have

$$(\bar{\nabla}_X\phi)U+(\bar{\nabla}_U\phi)X = -\alpha\eta(X)U-\beta\eta(X)PU-\beta\eta(X)FU. \tag{3.22}$$

Applying (2.4) in the left hand side of (3.22) and using (2.9), (2.5) and (2.13), it becomes

$$h(X,PU)-A_{FU}X+\nabla^\perp_X FU-\phi X(\ln f)U-X(\ln f)PU-2X(\ln f)FU - 2\phi h(X,U)+h(U,\phi X) = -\alpha\eta(X)U-\beta\eta(X)PU-\beta\eta(X)FU. \tag{3.23}$$

Taking the Riemannian metric g with $V \in \Gamma(TM_\theta)$ in (3.23), we derive

$$2g(h(U,X),FV) - g(h(X,V),FU) = (\phi X(\ln f) - \alpha\eta(X))\, g(U,V) + (X(\ln f) - \beta\eta(X))\, g(PU,V). \tag{3.24}$$

By switching U with V in (3.24), we have

$$2g(h(V,X),FU) - g(h(X,U),FV) = (\phi X(\ln f) - \alpha\eta(X))\, g(U,V) + (X(\ln f) - \beta\eta(X))\, g(PV,U). \tag{3.25}$$

From (3.24) and (3.25), we obtain (3.21), which proves the lemma. □

From Lemma 3.3, and by using (2.11), we derive the following equations. That is, by interchanging X by ϕX, U by PU and V by PV in (3.21), we get, respectively,

$$g(h(U,\phi X),FV) = -(X(\ln f) - \alpha\eta(\phi X))\, g(U,V) - \frac{1}{3}(\phi X(\ln f) - \beta\eta(\phi X))\, g(PV,U), \tag{3.26}$$

$$g(h(PU,X),FV) = (\phi X(\ln f) - \alpha\eta(X))\, g(PU,V) - \frac{1}{3}(X(\ln f) - \beta\eta(X))\cos^2\theta g(V,U), \tag{3.27}$$

$$g(h(U,X),FPV) = (\phi X(\ln f) - \alpha\eta(X))\, g(U,PV) + \frac{1}{3}(X(\ln f) - \beta\eta(X))\cos^2\theta g(V,U). \tag{3.28}$$

Again, by interchanging U by PU (V by PV) in (3.26), we get, respectively,

$$g(h(PU,\phi X),FV) = -(X(\ln f) - \alpha\eta(\phi X))\, g(PU,V) - \frac{1}{3}(\phi X(\ln f) - \beta\eta(\phi X))\cos^2\theta g(V,U), \tag{3.29}$$

$$g(h(U,\phi X),FPV) = -(X(\ln f) - \alpha\eta(\phi X))\, g(U,PV) + \frac{1}{3}(\phi X(\ln f) - \beta\eta(\phi X))\cos^2\theta g(V,U). \tag{3.30}$$

From (3.29) and by changing V to PV, we have

$$g(h(PU,\phi X),FPV) = -(X(\ln f) - \alpha\eta(\phi X))\cos^2\theta g(U,V) + \frac{1}{3}(\phi X(\ln f) - \beta\eta(\phi X))\cos^2\theta g(PV,U). \tag{3.31}$$

From (3.28) and by changing U to PU, we have

$$g(h(PU,X),FPV) = (\phi X(\ln f) - \alpha\eta(X))\cos^2\theta g(U,V) - \frac{1}{3}(X(\ln f) - \beta\eta(X))\cos^2\theta g(PV,U). \tag{3.32}$$

Clearly, from (3.29) and (3.30), and from (3.27) and (3.28), we have, respectively,

$$g(h(PU,\phi X),FV) + g(h(U,\phi X),FPV) = 0, \tag{3.33}$$

$$g(h(PU,X),FV) + g(h(U,X),FPV) = 0. \tag{3.34}$$

One can also see by (3.21), if ξ is tangent to M_T, we have

$$g(h(U,\xi),FV) = -\alpha g(U,V). \tag{3.35}$$

4. Main theorem. Let $\bar{M}$ be a $(2m+1)$-dimensional nearly trans-Sasakian manifold of type (α,β) and $M = B \times_f M_\theta$ be a CR-slant warped product submanifold of dimension n, where $B = M_T \times M_\perp$ is a CR-product submanifold tangent to ξ with $\dim(M_T) = 2s$ and $\dim(M_\perp) = t$. The slant submanifold M_θ is of dimension $2l$. Since $TM = D_T \oplus D_\perp \oplus D_\theta \oplus \langle\xi\rangle$, we consider the following orthonormal bases: $\{E_1,\dots,E_s,E_{s+1} = \phi E_1,\dots,E_{2s} = \phi E_s, E_{2s+1} = \xi\}$ for D_T, $\{\bar{E}_1 = E_{2s+2},\dots,\bar{E}_t = E_{2s+1+t}\}$ for $D_\perp$ and $\{E_1^* = E_{2s+t+2},\dots,E_l^* = E_{2s+t+1+l}, E_{l+1}^* = \sec\theta PE_1^*,\dots,E_{2l}^* = \sec\theta PE_l^*\}$ for D_θ. The orthonormal basis in the normal bundle $T^\perp M$ of $\phi D_\perp$ is $\{E_{n+1} = \hat{E}_1 = \phi\bar{E}_1,\dots,E_{n+t} = \hat{E}_t = \phi\bar{E}_t\}$, the orthonormal basis FD_θ is $\{E_{n+t+1} = \hat{E}_{t+1} = \csc\theta FE_1^*,\dots,E_{n+t+l} = \hat{E}_{t+l} = \csc\theta FE_l^*,\dots,E_{n+t+l+1} = \hat{E}_{t+l+1} = \csc\theta\sec\theta FPE_1^*,\dots,E_{n+t+2l} = \hat{E}_{t+21} = \csc\theta\sec\theta FPE_l^*\}$ and the invariant normal subbundle υ is $\{E_{n+t+2l+1},\dots,E_{2m+1}\}$. We use these frames in the following theorem.

Theorem 4.1. *Let $\bar{M}$ be a $(2m+1)$-dimensional nearly trans-Sasakian manifold of type (α,β) and $M = B \times_f M_\theta$ be a $D_\perp \oplus D_\theta$-mixed geodesic warped product CR-slant submanifold of $\bar{M}$ of dimension n, where $B = M_T \times M_\perp$ is a CR-product submanifold and M_θ is a slant submanifold of $\bar{M}$, $\dim M_\theta = 2l$. Then, the second fundamental form of M satisfies the following inequalities:*

(i) *If the structure vector field ξ is tangent to M_T, we have*

$$\|h\|^2 \geq \frac{2l}{9}\cos^2\theta\|\operatorname{grad}_\perp(\ln f)\|^2 + 4l\left[\left(\csc^2\theta + \frac{1}{9}\cot^2\theta\right)\left(\|\operatorname{grad}_T(\ln f)\|^2 - \beta^2\right) + \alpha^2\right]. \tag{4.1}$$

(ii) *If the structure vector field ξ is tangent to $M_\perp$, we have*

$$\|h\|^2 \geq \frac{2l}{9}\cos^2\theta(\|\operatorname{grad}_\perp(\ln f)\|^2 - \beta^2) + 4l\left(\csc^2\theta + \frac{1}{9}\cot^2\theta\right)\|\operatorname{grad}_T(\ln f)\|^2. \tag{4.2}$$

The equality holds identically in (4.1) *and* (4.2)*, if M_T and $M_\perp$ are totally geodesic submanifolds of $\bar{M}$ and M_θ is a totally umbilical submanifold of $\bar{M}$. Moreover, M is also $D_T \oplus D_\perp$-mixed geodesic in $\bar{M}$ but not $D_T \oplus D_\theta$-mixed geodesic. Hence, M is not minimal of $\bar{M}$.*

Proof. From the definition of h, we have

$$\|h\|^2 = \sum_{r=n+1}^{n+t}\sum_{i,j=1}^{n} g(h(E_i,E_j),E_r)^2 + \sum_{r=n+t+1}^{n+t+2l}\sum_{i,j=1}^{n} g(h(E_i,E_j),E_r)^2 + \sum_{r=n+t+2t+1}^{2m+1}\sum_{i,j=1}^{n} g(h(E_i,E_j),E_r)^2. \tag{4.3}$$

By considering the first and second terms in the right hand side of (4.3), one can obtain the following

$$\begin{aligned}\|h\|^2 \geq{}& \sum_{r=1}^{t}\sum_{i,j=1}^{2s+1} g(h(E_i,E_j),\phi\bar{E}_r)^2 + \sum_{r=1}^{t}\sum_{i,j=1}^{t} g(h(\bar{E}_i,\bar{E}_j),\phi\bar{E}_r)^2 \\ &+ \sum_{r=1}^{t}\sum_{i,j=1}^{2l} g(h(E_i^*,E_j^*),\phi\bar{E}_r)^2 + 2\sum_{r=1}^{t}\sum_{i=1}^{2s+1}\sum_{j=1}^{t} g(h(E_i,\bar{E}_j),\phi\bar{E}_r)^2 \\ &+ 2\sum_{r=1}^{t}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(E_i,E_j^*),\phi\bar{E}_r)^2 + 2\sum_{r=1}^{t}\sum_{i=1}^{t}\sum_{j=1}^{2l} g(h(\bar{E}_i,E_j^*),\phi\bar{E}_r)^2 \\ &+ \sum_{r=t+1}^{t+2l}\sum_{i,j=1}^{2s+1} g(h(E_i,E_j),\hat{E}_r)^2 + \sum_{r=t+1}^{t+2l}\sum_{i,j=1}^{t} g(h(\bar{E}_i,\bar{E}_j),\hat{E}_r)^2 \\ &+ \sum_{r=t+1}^{t+2l}\sum_{i,j=1}^{2l} g(h(E_i^*,E_j^*),\hat{E}_r)^2 + 2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{t} g(h(E_i,\bar{E}_j),\hat{E}_r)^2 \\ &+ 2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(E_i,E_j^*),\hat{E}_r)^2 + 2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{t}\sum_{j=1}^{2l} g(h(\bar{E}_i,E_j^*),\hat{E}_r)^2.\end{aligned} \tag{4.4}$$

First, we deal with the term $2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(E_i,E_j^*),\hat{E}_r)^2$ in the right hand side of (4.4). By decomposing this term and using (3.33) and (3.34), we obtain that

$$\begin{aligned}2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(E_i,E_j^*),\hat{E}_r)^2 = {}& 2\sum_{i=1}^{s}\sum_{r,j=1}^{l} g(h(E_i,E_j^*),\csc\theta FE_r^*)^2 \\ &+ 2\sum_{i=1}^{s}\sum_{r,j=1}^{l} g(h(E_i,\sec\theta PE_j^*),\sec\theta\csc\theta FPE_r^*)^2\end{aligned} \tag{4.5}$$

$$+2\sum_{i=1}^{s}\sum_{r,j=1}^{l} g(h(\phi E_i, E_j^*), \csc\theta F E_r^*)^2$$
$$+2\sum_{i=1}^{s}\sum_{r,j=1}^{l} g(h(\phi E_i, \sec\theta P E_j^*), \sec\theta\csc\theta F P E_r^*)^2$$
$$+2\sum_{r=t+1}^{t+2l}\sum_{r=1}^{2l} g(h(\xi, E_j^*), \hat{E}_r)^2.$$

Using equations (3.21), (3.26), (3.31), (3.32) in (4.5) and since $\eta(E_i) = 0$ for any $i \in \{1, \dots, 2s\}$, we have

$$2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(E_i, E_j^*), \hat{E}_r)^2 = 2\sum_{r=t+1}^{t+2l}\sum_{r=1}^{2l} g(h(\xi, E_j^*), \hat{E}_r)^2 + 4l\left(\csc^2\theta + \frac{1}{9}\cot^2\theta\right)\left[\sum_{i=1}^{2s+1}(E_i(\ln f))^2 - (\xi(\ln f))^2\right]. \tag{4.6}$$

By applying (2.7), (3.35) and since $\xi(\ln f) = \beta$, we can rewrite (4.6) as

$$2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(E_i, E_j^*), \hat{E}_r)^2 = 4l\left[\left(\csc^2\theta + \frac{1}{9}\cot^2\theta\right)(\|\operatorname{grad}_T(\ln f)\|^2 - \beta^2) + \alpha^2\right]. \tag{4.7}$$

Now, we decompose the term $\sum_{r=1}^{t}\sum_{i,j=1}^{2l} g(h(E_i^*, E_j^*), \phi\bar{E}_r)^2$ in (4.4) as follows

$$\sum_{r=1}^{t}\sum_{i,j=1}^{2l} g(h(E_i^*, E_j^*), \phi\bar{E}_r)^2 = \sum_{r=1}^{t}\sum_{i,j=1}^{l}[g(h(E_i^*, E_j^*), \phi\bar{E}_r)^2 + g(h(\sec\theta P E_i^*, E_j^*), \phi\bar{E}_r)^2 + g(h(E_i^*, \sec\theta P E_j^*), \phi\bar{E}_r)^2 + g(h(\sec\theta P E_i^*, \sec\theta P E_j^*), \phi\bar{E}_r)^2]. \tag{4.8}$$

Since M is $D_\perp \oplus D_\theta$-mixed geodesic and by (3.20), the second and third terms in (4.8) vanish. Using (3.14), (3.19) and since $\eta(\bar{E}_i) = 0$ for any $r \in \{1, \dots, t\}$, we obtain

$$\sum_{r=1}^{t}\sum_{i,j=1}^{2l} g(h(E_i^*, E_j^*), \phi\bar{E}_r)^2 = \frac{2l}{9}\cos^2\theta\sum_{r=1}^{t}\left(\bar{E}_r(\ln f)\right)^2 = \frac{2l}{9}\cos^2\theta\|\operatorname{grad}_\perp(\ln f)\|^2. \tag{4.9}$$

Hence, from (4.7) and (4.9), we get (4.1), which proves the inequality (i).

Now, to prove (ii), let ξ be tangent to $M_\perp$. From (4.6) and by using (2.7), we get

$$2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(E_i,E_j^*),\hat{E}_r)^2 = 4l(\csc^2\theta+\frac{1}{9}\cot^2\theta)\|\operatorname{grad}_T(\ln f)\|^2. \tag{4.10}$$

Also, form (4.9) and by (2.7), we get

$$\begin{aligned}\sum_{r=1}^{t}\sum_{i,j=1}^{2l} g(h(E_i^*,E_j^*),\phi\bar{E}_r)^2 &= \frac{2l}{9}\cos^2\theta\left[\sum_{r=1}^{t+1}\big(\bar{E}_r(\ln f)\big)^2-(\xi(\ln f))^2\right]\\ &= \frac{2l}{9}\cos^2\theta(\|\operatorname{grad}_\perp(\ln f)\|^2-\beta^2).\end{aligned} \tag{4.11}$$

Hence, from (4.10) and (4.11), we get (4.2), which proves the inequality (ii).

If equality holds in (4.1) and (4.2), then from (4.3) and since M is $D_\perp\oplus D_\theta$-mixed geodesic in $\bar{M}$, we find

$$\begin{gathered}h(D_T,D_T)=0,\quad h(D_\perp,D_\perp)=0,\quad h(D_\theta,D_\theta)\subseteq D_\perp,\\ h(D_T,D_\perp)=0,\quad \text{and}\quad h(D_\perp,D_\theta)=0.\end{gathered} \tag{4.12}$$

From (3.6) and by (4.12), we vanish the fifth term of the right hand side of (4.4). Thus, from the above facts and since B is totally geodesic and M_θ is totally umbilical in M, we deduce that M_T and $M_\perp$ are totally geodesic submanifolds of $\bar{M}$, while M_θ is a totally umbilical submanifold of $\bar{M}$. Moreover, from (4.12), M is also $D_T\oplus D_\perp$-mixed geodesic. M can never be a $D_T\oplus D_\theta$-mixed geodesic, which proves the theorem. □

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