

ON 1-ABSORBING PRIME AND 1-ABSORBING PRIMARY
IDEALS OF AMALGAMATIONS

Abuzer Gündüz

Received on May 17, 2023
Presented by V. Drensky, Member of BAS, on June 27, 2023

Abstract

In this note, we define the 1-*absorbing* prime ideals and 1-*absorbing* primary ideals in the amalgamations of A with B along J with respect to f denoted $A \bowtie^f J$ and give some results.

Key words: 1-absorbing prime ideal, 1-absorbing primary ideal, amalgamated algebra along an ideal

2020 Mathematics Subject Classification: 13A99, 13C13

1. Introduction. We assume throughout that all rings are commutative with $1 \neq 0$ and all modules are assumed to be unital. Among the many generalizations of the prime ideals in the literature, we focus on the 1-absorbing prime ideals and 1-absorbing primary ideals, which were studied in [1] and [2], respectively. A proper ideal I is called 1-absorbing prime if for all non-unit elements $a, b, c \in R$ such that $abc \in I$, then either $ab \in I$ or $c \in I$. Some properties of 1-absorbing prime ideal were studied in [1]. For instance, they showed if I is a 1-absorbing prime ideal of R , then $\sqrt{I}$ is a prime ideal of R . If R admits 1-absorbing prime ideal which is not a prime ideal, then R is a quasi-local ring. The equivalent condition to the definition of 1-absorbing prime ideal with the ideals of R was proved. Also, they gave the 1-absorbing prime avoidance theorem for the 1-absorbing prime ideal. On the other hand, a proper ideal I is called 1-absorbing primary if for all non-unit elements $a, b, c \in R$ such that $abc \in I$, then either $ab \in I$ or $c \in \sqrt{I}$.

Some properties of 1-absorbing primary ideal were studied in [2]. Firstly, they showed the following implications hold, and none of them is reversible:

$$\text{Primary ideal} \Rightarrow \text{1-absorbing primary ideal} \Rightarrow \text{2-absorbing primary ideal}$$

The authors (in [2]) define the 1-absorbing primary ideal of R that is not a primary ideal of R , and a 2-absorbing primary (studied by BADAWI et al. in [3]) ideal of R that is not 1-absorbing primary ideal of R . They showed (in [2]) if I is a 1-absorbing primary ideal of R , then $\sqrt{I}$ is a prime ideal of R and if R admits a 1-absorbing primary ideal of R which is not a primary ideal, then R is a quasilocal ring. A method is also given to construct 1-absorbing primary ideals of commutative rings that are not primary ideals. Also, they gave an equivalent condition with ideals to the definition of 1-absorbing primary ideal.

Let A and B be two rings, J be an ideal of B , and $f : A \rightarrow B$ be a ring homomorphism. Meanwhile, let us take the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) : a \in A, j \in J\}$$

which is named the amalgamation of A and B along J with respect to f . This construction is a generalization of the amalgamated duplication of a ring along an ideal denoted by $A \bowtie J$, which was introduced and studied by D'ANNA and FONTANA in [4]. In [5,6], they studied amalgamations in the frame of pullbacks. Many basic properties of various ideal and ring-theoretic can be obtained with the help of this method. This topic is a significant and interesting concept studied by many well-known algebraists. This concept of amalgamation resides in their ability to cover basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations).

Besides, constructions such as the $A + XB[X]$, the $A + XB[[X]]$ and the $D + M$ can be studied as particular cases of the amalgamation ([5], Examples 2.5 and 2.6) and other classical constructions, such as CPI extensions (in the sense of BOISEN and SHELDON [7]) are definitely related to it ([5], Example 2.7 and Remark 2.8). In [5], the authors studied the basic properties of this construction such as characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, and a reduced ring. Also, they characterized those distinguished pullbacks that can be expressed as an amalgamation.

Furthermore, in [6] they studied the structure of the ring of the form $A \bowtie^f J$, with certain caution to the prime spectrum to the chain properties and to the Krull dimension.

In [8], the authors gave the structure of the ring of the form $A \bowtie^f J$, with particular attention to the form of weakly prime ideals and too weakly Krull dimension.

In [9], the authors studied n -absorbing ideals and strongly n -absorbing ideals in the amalgamation of A with B along J with respect to f denoted by $A \bowtie^f J$

and gave some results. For a ring they described $\text{Spec}(R)$, (resp. $\text{Max}(R)$) were the set of all prime (resp. Maximal) ideals of R not containing I . Moreover, they denoted by $\text{Nil}(R)$ (resp. $\text{Jac}(R)$), the set of nilpotent elements of R (resp. the Jacobson radical of R). In [10], they studied over 1-absorbing prime ideals and 1-absorbing primary ideals of amalgamations through their definitions. But we studied by using isomorphism in ([5], Proposition 5.1.(2)) like [9]. Also, our main results Theorem 1 and Theorem 2, are more precise because they are unconditional. However, their main theorems depend on some conditions such as $A \bowtie^f J$ be a local ring, I is not primary. In this note, we define the 1-absorbing prime ideals and 1-absorbing primary ideals in the amalgamations of A with B along J with respect to f denoted $A \bowtie^f J$ with some results. For example, we give the 1-absorbing prime ideals of $A \bowtie^f J$ when A is a Noetherian integral domain which is not a field, a quasi-local ring, localization of ring and 1-absorbing primary ideals of $A \bowtie^f J$ when A is a valuation domain, Prufer domain, Noetherian integral domain, a Dedekind domain, and localization of ring.

2. 1-absorbing prime ideals of amalgamations. To avoid repetition, let us use the notation for the rest of the paper. Let A and B be rings and $f : A \rightarrow B$ be a ring homomorphism, and J be an ideal of B . All along the paper, $A \bowtie^f J$ denotes the amalgamation of A with B along J with respect to f . Let I be an ideal of A and K be an ideal of $f(A) + J$. Let us define the sets:

$$I \bowtie^f J := \{(i, f(i) + j) : i \in I, j \in J\}$$

$$\overline{K}^f := \{(a, f(a) + j) : a \in A, j \in J, f(a) + j \in K\}$$

$$\overline{I \times K}^f := \{(a, f(a) + j) : a \in I, j \in J, f(a) + j \in K\}.$$

It is clear that $I \bowtie^f J$, $\overline{K}^f$, $\overline{I \times K}^f$ are ideals of $A \bowtie^f J$. Our first result explains when the ideals $I \bowtie^f J$, $\overline{K}^f$ and $\overline{I \times K}^f$ are 1-absorbing prime ideals of $A \bowtie^f J$. Besides we will study non-unit elements $a, b, c \in R$ along this note.

With some conditions, the following theorem is given in ([10], Theorem 2.1). However, we will give its unconditional and prove by isomorphism in [5].

Theorem 1. *Under the above notations, the following statements hold:*

- 1) $I \bowtie^f J$ is a 1-absorbing prime ideal of $A \bowtie^f J$ if and only if I is 1-absorbing prime ideal of A .
- 2) If I is 1-absorbing prime ideal of A and K is 1-absorbing prime ideal of $f(A) + J$, then $\overline{I \times K}^f$ is a 1-absorbing prime ideal of $A \bowtie^f J$.
- 3) $\overline{K}^f$ is a 1-absorbing prime ideal of $A \bowtie^f J$ if and only if K is a 1-absorbing prime ideal of $f(A) + J$.

To prove Theorem 1, we shall give the following lemmas, respectively.

Lemma 1 ([1]). *Under the above notations with $f(1) = 1$ and $f(a)$ being non-unit element in B , the following statements hold:*

1) If J is a 1-absorbing prime ideal of B , then $f^{-1}(J)$ is a 1-absorbing prime ideal of A .

2) Suppose that f is surjective and I is an ideal of A containing $\ker(f)$. Then I is a 1-absorbing prime ideal of A if and only if $f(I)$ is a 1-absorbing prime ideal of B .

Proof. First part is proved in ([1], Theorem 2.16(1)).

We will prove only the second part of this lemma.

$\Rightarrow$: it done in [1] Theorem 2.16.(2).

$\Leftarrow$: Let us take $I = \ker(f)$ and $J = I$ in ([1], Corollary 2.17), we obtain $I/\ker(f)$ is 1-absorbing prime ideal of $A/\ker(f)$. On the other hand, since $A/\ker(f) \cong B$ and $I/\ker(f)$ is 1-absorbing prime ideal of B , then by Lemma 1.(1) we get $f^{-1}(I/\ker(f)) = I$ is 1-absorbing prime ideal of A , as desired. $\square$

Lemma 2. Under the above notations, the following hold:

1) Let $A \subseteq B$ be an extension of rings and J be a 1-absorbing prime ideal of B . Then $A \cap J$ is a 1-absorbing prime ideal of A .

2) Assume that I and J are proper ideals of A with $I \subseteq J$ and $U(R/I) = \{a + I : a \in U(R)\}$. Then J is a 1-absorbing prime ideal of A if and only if J/I is a 1-absorbing prime ideal of A/I . Especially, I is a 1-absorbing prime ideal of A if and only if $\{0\}$ is a 1-absorbing prime ideal of A/I .

Proof. 1) The statement is directly corollary of Lemma 1(1), if we can see $f^{-1}(J) = A \cap J$.

2) By ([1], Corollary 2.17), we get the statement's first part and take especially $I = J$; we hold the second statement of the lemma. $\square$

Proof of Theorem 1. Recall that $\frac{A \bowtie^f J}{I \bowtie^f J} \cong \frac{A}{I}$ ([5], Proposition 5.1.(2)).

Consider the ring homomorphism $v : \frac{A \bowtie^f J}{I \bowtie^f J} \rightarrow \frac{A}{I}$.

1) By assertion of Lemma 2(2), I is a 1-absorbing prime ideal of A if and only if $\{0\}$ is ideal of $\frac{A}{I}$ and by isomorphism if and only if $\{0\}$ is ideal of $\frac{A \bowtie^f J}{I \bowtie^f J}$ and assertion of Lemma 2(2), $I \bowtie^f J$ is ideal of $A \bowtie^f J$, as desired.

2) Let a_i be non-unit elements of A and $j_i \in J$ and so $f(a_i) + j \in K$ for $i = 1, 2, 3$. Assume that $(a_1, f(a_1) + j_1)(a_2, f(a_2) + j_2)(a_3, f(a_3) + j_3) \in \overline{I \times K}^f$ and implies $(a_1 a_2, j_2 f(a_1) + j_1 f(a_2) + f(a_1) f(a_2) + j_1 j_2)(a_3, f(a_3) + j_3) \in \overline{I \times K}^f$. Since f is a ring homomorphism we get $(a_1 a_2, f(j_2 a_1 + j_1 a_2 + a_1 a_2) + j_1 j_2)(a_3, f(a_3) + j_3) \in \overline{I \times K}^f$. On the other hand, since I is an 1-absorbing prime ideal of A , if $a_1 a_2 a_3 \in I$, then either $a_1 a_2 \in I$ or $a_3 \in I$. If $a_1 a_2 \in I$, since $j_2 a_1 + j_1 a_2 + a_1 a_2 \in A$ and $j_1 j_2 \in J$, then $(a_1 a_2, f(j_2 a_1 + j_1 a_2 + a_1 a_2) + j_1 j_2) \in \overline{I \times K}^f$. If $a_3 \in I$, then $(a_3, f(a_3) + j_3) \in \overline{I \times K}^f$, as desired.

3) Let K be 1-absorbing prime ideal of $f(A) + J$. Let us define the map $p_\gamma : A \bowtie^f J \rightarrow f(A) + J$ with $(a, f(a) + j) \mapsto f(a) + j$. Clearly, p_γ is surjective

and $\text{Ker}(p_\gamma) = A \times \{0\} \subseteq \overline{K}^f$ and $p_\gamma(\overline{K}^f) = K$. Hence by Lemma 1 (2), we have the desired result. $\square$

Corollary 1. *Let $f : A \rightarrow B$ be a ring homomorphism, J be an ideal of B , and I, P, M be an ideal of A . Consider a positive integer n . Then the following assertions hold:*

1) *Let A be a Noetherian integral domain which is not field and I an ideal of A . Moreover if A is a Dedekind domain, then $I \bowtie^f J$ is 1-absorbing prime ideal of $A \bowtie^f J$. Also in this case $I = P$ or $I = P^2$, where $p = \sqrt{I}$ is a prime ideal of A .*

2) *If A is quasi-local π -regular with a maximal ideal $M = \text{Nil}(A) \neq \{0\}$ such that $(M^2 : a) \subseteq rx$ for each nonzero $x \in M$, $\forall r \in A$ and $a \in M$, (x) and I is any nonzero proper ideal of A , then $I \bowtie^f J$ is 1-absorbing prime ideal of $A \bowtie^f J$.*

3) *Assume that $\text{Nil}(A)$ and P be divided prime ideals of A . Then $P^2 \bowtie^f J$ is 1-absorbing prime ideal of $A \bowtie^f J$.*

4) *Let A be a ring, I be proper ideal of A and S be a multiplicative closed subset such that $I \cap S = \emptyset$. If I is 1-absorbing prime ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I \bowtie^f J$ is 1-absorbing prime ideal of $S^{-1}A \bowtie^f J$.*

5) *Let M be an A -module and $I(+)N$ be a homogenous ideal of the ring $A(+)M$. If $I(+)N$ is a 1-absorbing prime ideal of $A(+)M$, then $I \bowtie^f J$ is 1-absorbing prime ideal of $A \bowtie^f J$.*

6) *Let A be a ring and I be a P -primary ideal of A . If $(P^2 : x) \subseteq I$ for every $x \in P \setminus I$, then $I \bowtie^f J$ is 1-absorbing prime ideal of $A \bowtie^f J$.*

Proof. The proof 1–6 of Corollary 1 can be proved using ([¹], Theorem 2.11), ([¹], Theorem 2.14), ([¹], Theorem 2.15), ([¹], Theorem 2.18), ([¹], Theorem 2.20), ([¹], Theorem 2.13), respectively. $\square$

The next Corollary is given in ([¹⁰], Corollary 2.11) with the condition that T and D are local. However, we will not give any conditions.

Corollary 2. *Let M be a maximal ideal of an integral domain T and D a subring of T such that $D \cap M = \{0\}$. Let I be an ideal of D . Then $I + M$ is a 1-absorbing prime ideal of $D + M$ if and only if I is 1-absorbing prime ideal of D .*

Proof. Consider $\sigma : D \hookrightarrow T$ natural embedding. Since ([⁵], Proposition 5), $D \bowtie^f M \cong D + M$ (as $D \cap M = \{0\}$) and $I \bowtie^f M \cong I + M$, then $I + M$ is a 1-absorbing prime ideal of $D + M$ if and only if $I \bowtie^f M$ is a 1-absorbing prime ideal of $D \bowtie^f M$. By assertion (1) of Theorem 1, it implies that $I + M$ is a 1-absorbing prime ideal of $D + M$ if and only if I is a 1-absorbing prime ideal of D . $\square$

Example 1. Let $A = \mathbb{Z}$ be a ring of integers, $B = \mathbb{Q}[[X]]$ be a ring of formal power series over $\mathbb{Q}$ in an indeterminate X , $f : \mathbb{Z} \rightarrow \mathbb{Q}[[X]]$ be the natural embedding and $J = X\mathbb{Q}[[X]]$ be the maximal ideal of B . Let $H = \{XP(X) : P \in f(A) + J = \mathbb{Z} + X\mathbb{Q}[[X]]\}$. Obviously, H is an ideal of $f(A) + J = \mathbb{Z} + X\mathbb{Q}[[X]]$ and $0 \bowtie^f H$ is an ideal of $\mathbb{Z} \bowtie^f \mathbb{Q}[[X]]$. We claim that $0 \bowtie^f H$ is not 1-absorbing prime ideal of $\mathbb{Z} \bowtie^f \mathbb{Q}[[X]]$. Indeed, $0 \bowtie^f H = \{(0, XP(X)) : p(X) \in \mathbb{Z} + X\mathbb{Q}[[X]]\}$

and we have $(3, 3)^2 \cdot \left(0, \frac{X}{3^2}\right) = (0, X) \in 0 \bowtie^f H$. But $(3, 3)^2 \notin 0 \bowtie^f H$ and $\left(0, \frac{X}{3^2}\right) \notin 0 \bowtie^f H$, as desired.

To another counter-example of 1-absorbing prime ideal of amalgamations, we consider ([1], Example 2.2), with our Theorem 1.

Example 2. Let $A = \mathbb{Z} + 5x\mathbb{Z}[x]$, $I = 5x\mathbb{Z}[x]$, $B = \mathbb{Z}[x]$, $J = x\mathbb{Z}[x]$ and $f : A \rightarrow B$. We can see that I^2 is not a 1-absorbing prime ideal of A . Because, $(5)(5)(x^2) \in I^2$, but neither $25 \notin I^2$ or $x^2 \notin I^2$. So, by Theorem 1(1), we get $I \bowtie^f J$ is not a 1-absorbing prime ideal of $A \bowtie^f J$.

3. 1-absorbing primary of amalgamations. In this section, we will define the 1-absorbing primary ideals of $A \bowtie^f J$ and give some results. Recall that a proper ideal I is called 1-absorbing primary if for all non-unit elements $a, b, c \in R$ such that $abc \in I$, then either $ab \in I$ or $c \in \sqrt{I}$. The next theorem characterizes the form of 1-absorbing primary ideals $A \bowtie^f J$ in some particular cases. We will use the same definitions given in the previous section:

With some conditions, the following theorem is given in ([10], Theorem 2.9). However, we will give its unconditional and prove by isomorphism in [5].

Theorem 2. *Under the above notations, the following statements hold:*

- 1) $I \bowtie^f J$ is a 1-absorbing primary ideal of $A \bowtie^f J$ if and only if I is 1-absorbing primary ideal of A .
- 2) If I is 1-absorbing primary ideal of A and K is 1-absorbing primary ideal of $f(A) + J$, then $\overline{I \times K}^f$ is a 1-absorbing primary ideal of $A \bowtie^f J$.
- 3) $\overline{K}^f$ is a 1-absorbing primary ideal of $A \bowtie^f J$ if and only if K is a 1-absorbing primary ideal of $f(A) + J$.

To prove Theorem 2, we shall give the following lemmas, respectively.

Lemma 3 ([2]). *Under the above notations, the following statements hold:*

- 1) If J is a 1-absorbing primary ideal of B , then $f(J)^{-1}$ is a 1-absorbing primary ideal of A .
- 2) Suppose that f is surjective and I is a 1-absorbing primary ideal of A containing $\text{Ker}(f)$. Then $f(I)$ is a 1-absorbing primary ideal of B if and only if I is a 1-absorbing primary ideal of A .

Proof. First part is proved in ([2], Theorem 2.16(1)).

We will prove only the second part of this lemma.

$\Leftarrow$: it done in [2] Theorem 17(2).

$\Rightarrow$: Let us take $I = \text{Ker}(f)$ and $J = I$ in ([2], Corollary 3), we obtain $I/\text{Ker}(f)$ is 1-absorbing primary ideal of $A/\text{Ker}(f)$ and since $A/\text{Ker}(f) \cong B$ and $I/\text{Ker}(f)$ is 1-absorbing primary ideal of B , then by Lemma 4(1) we get $f^{-1}(I/\text{Ker}(f)) = I$ is 1-absorbing primary ideal of A , as desired. $\square$

Lemma 4. *Under the above notations, the following hold:*

- 1) Let $A \subseteq B$ be an extension of rings and J be a 1-absorbing primary ideal of B . Then $A \cap J$ is a 1-absorbing primary ideal of A .

2) Assume that I and J are proper ideals of A with $I \subseteq J$. Then J is a 1-absorbing primary ideal of A if and only if J/I is a 1-absorbing primary ideal of A/I . Especially, I is a 1-absorbing primary ideal of A if and only if $\{0\}$ is a 1-absorbing primary ideal of A/I .

Proof. 1) The statement is directly corollary of Lemma 3(1), if we can see $f^{-1}(J) = A \cap J$.

2) By ([2], Corollary 3), we get the statement's first part and take especially $I = J$, we hold the second statement of the lemma. $\square$

Proof of Theorem 2. Recall that $\frac{A \bowtie^f J}{I \bowtie^f J} \cong \frac{A}{I}$ ([5], Proposition 5.1.(2)).

Consider the ring homomorphism $v : \frac{A \bowtie^f J}{I \bowtie^f J} \rightarrow \frac{A}{I}$.

1) By assertion (2) of Lemma 4, I is a 1-absorbing prime ideal of A if and only if $\{0\}$ is ideal of $\frac{A}{I}$ and by isomorphism if and only if $\{0\}$ is ideal of $\frac{A \bowtie^f J}{I \bowtie^f J}$ and again assertion (2) of Lemma 4 $I \bowtie^f J$ is ideal of $A \bowtie^f J$, as desired.

2) Let a_i be non-unit elements of A and $j_i \in J$ and so $f(a_i) + j \in K$ for $i = 1, 2, 3$. Assume that $(a_1, f(a_1) + j_1)(a_2, f(a_2) + j_2)(a_3, f(a_3) + j_3) \in \overline{I \times K^f}$ and implies $(a_1 a_2, j_2 f(a_1) + j_1 f(a_2) + f(a_1) f(a_2) + j_1 j_2)(a_3, f(a_3) + j_3) \in \overline{I \times K^f}$. Since f is a ring homomorphism we get $(a_1 a_2, f(j_2 a_1 + j_1 a_2 + a_1 a_2) + j_1 j_2)(a_3, f(a_3) + j_3) \in \overline{I \times K^f}$. On the other hand, since I is a 1-absorbing primary ideal of A , if $a_1 a_2 a_3 \in I$, then either $a_1 a_2 \in I$ or $a_3^n \in I$, for some $n \geq 1$. If $a_1 a_2 \in I$, since $j_2 a_1 + j_1 a_2 + a_1 a_2 \in A$ and $j_1 j_2 \in J$, then $(a_1 a_2, f(j_2 a_1 + j_1 a_2 + a_1 a_2) + j_1 j_2) \in \overline{I \times K^f}$. If $a_3^n \in I$, then $(a_3^n, f(a_3^n) + j_3) \in \overline{I \times K^f}$, as desired.

3) Let K be a 1-absorbing primary ideal of $f(A) + J$. Let us define the map $p_\gamma : A \bowtie^f J \rightarrow f(A) + J$ with $(a, f(a) + j) \mapsto f(a) + j$. Clearly, p_γ is surjective, $\ker(p_\gamma) = A \times \{0\} \subseteq \overline{K^f}$ and $p_\gamma(\overline{K^f}) = K$. Hence by Lemma 3(2), we have the desired result. $\square$

To counter-example of 1-absorbing primary ideal of amalgamations, we consider ([2], Example 2) with our Theorem 2(1).

Example 3. Let $A = \mathbb{Z}$, $I = 12\mathbb{Z}$, $B = \mathbb{Z}$, $J = \mathbb{Z}$ and $f : A \rightarrow B$. We can see that I is not a 1-absorbing primary ideal of A . Because, $(2)(2)(3) \in I$, but neither $(2)(2) \notin I$ nor $3 \notin \sqrt{I}$. So, by Theorem 2(1), we get $I \bowtie^f J$ is not an 1-absorbing primary ideal of $A \bowtie^f J$.

Corollary 3. Let $f : A \rightarrow B$ be a ring homomorphism, J be an ideal of B , and I, P, M be an ideal of A and a positive integer n . Then the following statements hold:

1) If I is an primary ideal of A , then $I \bowtie^f J$ is 1-absorbing primary ideal of $A \bowtie^f J$.

2) Let A be a quasi-local ring with maximal ideal M and $x \in M$ a nonzero prime element of A such that $M \neq xR$ and $I = xM$. Then $I \bowtie^f J$ is 1-absorbing

primary ideal of $A \bowtie^f J$.

3) Let A be a quasi-local ring with maximal ideal M and P be a prime ideal of A such that $P \subseteq M$. Then $PM \bowtie^f J$ is 1-absorbing primary ideal of $A \bowtie^f J$.

4) Let A be a divided integral domain and P be a prime ideal of A . Then, for $n \geq 1$, $P^n \bowtie^f J$ is 1-absorbing primary ideal of $A \bowtie^f J$.

5) Let A be a valuation domain and I be a proper ideal of A with $\sqrt{I} = P$, where P is a prime ideal of A . If I is primary ideal of A , then $I \bowtie^f J$ is 1-absorbing primary ideal of $A \bowtie^f J$.

6) Let A be a Prufer domain and I be a proper ideal of A with $\sqrt{I} = P$ for some prime ideal P of A such that P is invertible (i.e. P is finitely generated ideal of A). If I is primary ideal of A , then $I \bowtie^f J$ is 1-absorbing primary ideal of $A \bowtie^f J$.

7) Let A be a Dedekind domain and I be a proper nonzero ideal of A . If $\sqrt{I}$ is a prime ideal of A , then $I \bowtie^f J$ is 1-absorbing primary ideal of $A \bowtie^f J$.

8) Let A be a Noetherian integral domain that is not a field and a Dedekind domain. If $I = P^n$ for some prime ideal P of A for some positive integer n , then $I \bowtie^f J$ is 1-absorbing primary ideal of $A \bowtie^f J$.

9) Let S be multiplicatively closed subset of A and I be a proper ideal of A such that $I \cap S = \emptyset$. $I \bowtie^f J$ is 1-absorbing primary ideal of $A \bowtie^f J$ if and only if $S^{-1}I \bowtie^f J$ is 1-absorbing primary ideal of $S^{-1}A \bowtie^f J$.

Proof. The proof 1 – 9 of Corollary 3 can be proved using ([2], Theorem 1), ([2], Theorem 5), ([2], Theorem 7), ([2], Theorem 11), ([2], Theorem 12), ([2], Theorem 14), ([2], Theorem 15), ([2], Theorem 18), respectively. $\square$

4. Conclusion. In this note, we define the 1-absorbing prime ideals and 1-absorbing primary ideals in the amalgamations of A with B along J with respect to f denoted $A \bowtie^f J$ with some results. Compared to [10], our main theorems and conclusions are more precise because our results do not depend on any conditions. We also use a different method which is isomorphisms to obtain them. In general, we give the 1-absorbing prime ideals of $A \bowtie^f J$ when A is a Noetherian integral domain which is not a field, a quasi-local ring, localization of ring and 1-absorbing primary ideals of $A \bowtie^f J$ when A is a valuation domain, Prufer domain, Noetherian integral domain, a Dedekind domain, and localization of ring.

REFERENCES

- [1] YASSINE A., M. J. NIKMEHR, R. NIKANDISH (2021) On 1-absorbing prime ideals of commutative rings, J. Algebra Appl., **20**(10), 2150175.
- [2] BADAWI A., E. Y. ÇELIKEL (2020) On 1-absorbing primary ideals of commutative rings, J. Algebra Appl., **19**(06), 2050111.
- [3] BADAWI A., U. TEKIR, E. YETKIN (2014) On 2-absorbing primary ideals in commutative rings, Bull. Korean Math. Soc, **51**(4), 1163–1173.

- [4] D'ANNA M., M. FONTANA (2007) An amalgamated duplication of a ring along an ideal: the basic properties, *J. Algebra Appl.*, **6**(03), 443–459.
- [5] D'ANNA M., C. A. FINOCCHIARO, M. FONTANA (2009) Amalgamated algebras along an ideal, *Walter de Gruyter GmbH & Co. KG, Berlin*, 155–172.
- [6] D'ANNA M., C. A. FINOCCHIARO, M. FONTANA (2010) Properties of chains of prime ideals in an amalgamated algebra along an ideal, *J. Pure Appl. Algebra*, **214**(9), 1633–1641.
- [7] BOISEN M. B., P. B. SHELDON (1977) CPI-extensions: overrings of integral domains with unique prime spectrums, *Canadian J. Math.*, **29**(4), 722–737.
- [8] MAHDOU N., M. A. S. MOUTUI, Y. ZAHIR (2020) Weakly prime ideals issued from an amalgamated algebra, *Hacet. J. Math. Stat.*, **49**(3), 1159–1167.
- [9] ISSOUAL M., N. MAHDOU, M. A. S. MOUTUI (2020) On n -absorbing and strongly n -absorbing ideals of amalgamation, *J. Algebra Appl.*, **19**(10), 2050199.
- [10] KHALFI A. E., T. KOLOTOĞLU, N. MAHDOU, Ü. TEKİR, B. A. ERSOY (2023) Amalgamated algebras along an ideal defined by 1-absorbing-like conditions, *J. Algebra Appl.*, 2450090, <https://doi.org/10.1142/S0219498824500907>.

Department of Mathematics
Sakarya University
54050 Sakarya, Turkey
e-mail: abuzergunduz@sakarya.edu.tr