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NOETHER'S PROBLEM FOR ABELIAN EXTENSIONS OF CYCLIC P-GROUPS OF NILPOTENCY CLASS 2

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Abstract

Let K be a field and G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K-automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. Noether's problem then asks whether $K(G)$ is rational over K. Let p be prime and let G be a p-group of exponent p^e. Assume also that (i) char $K = p > 0$, or (ii) char $K \neq p$ and K contains a primitive p^e-th root of unity. In this paper we prove that $K(G)$ is rational over K if G is any finite p-group of nilpotency class 2 which is an abelian extension of a cyclic group.

Key words: Noether's problem, the rationality problem, nilpotent groups, p-groups, metabelian groups

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1. Introduction. Let K be a field and G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K-automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. *Noether's problem* then asks whether $K(G)$ is rational (= purely transcendental) over K. It is related to the inverse Galois problem, to the existence of generic G-Galois extensions over k, and to the existence of versal G-torsors over k-rational field extensions (see [8,9] and ([3], 33.1, p. 86)). Noether's problem for abelian groups was studied extensively by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. Fischer's Theorem is a starting point of investigating Noether's problem for finite abelian groups in general.

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Theorem 1.1 (Fischer [9], Theorem 6.1)**.** *Let G be a finite abelian group of exponent e. Assume that* (i) *either char $K = 0$ or char $K > 0$ with char $K \nmid e$, and* (ii) *K contains a primitive e-th root of unity. Then $K(G)$ is rational over K.*

Swan's paper [9] also gives a survey of many results related to the Noether's problem for abelian groups. On the other hand, the systematic study of Noether's problem for non-abelian groups is a more difficult process and it is far from complete.

Definition 1.1. We say that a group G has the AEC (Abelian Extension of a Cyclic group) property if G has a normal abelian subgroup H such that the quotient group G/H is cyclic.

Recently the first author of the present article gave a positive answer to Noether's problem in [7] for several p-groups G of nilpotency class 2, defined by specific generators and relations. However, Noether's problem for p-groups with the AEC property is still not solved entirely. In the main result of this paper, we give a positive answer to Noether's problem for any p-group G of nilpotency class 2, which has the AEC property. In Section 3 we prove:

Theorem 1.2. *For any prime p let G be a p-group of nilpotency class 2, which has the AEC property. Denote by p^e the exponent of G. Assume that* (i) *char $K = p > 0$, or* (ii) *char $K \neq p$ and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

The key idea to prove our results is to find a faithful G-subspace W of the regular representation space $\bigoplus_{g \in G} K \cdot x(g)$ and to show that W^G is rational over K. The subspace W is obtained as an induced representation from H.

2. Generalities. We list several results which will be used in the sequel.

Theorem 2.1 ([4], Theorem 1)**.** *Let G be a finite group acting on $L(x_1, \ldots, x_m)$, the rational function field of m variables over a field L such that*

(i) *for any $\sigma \in G, \sigma(L) \subset L$;*

(ii) *the restriction of the action of G to L is faithful;*

(iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + B(\sigma),$$

where $A(\sigma) \in \mathrm{GL}_m(L)$ and $B(\sigma)$ is $m \times 1$ matrix over L. Then there exist $z_1, \ldots, z_m \in L(x_1, \ldots, x_m)$ so that $L(x_1, \ldots, x_m)^G = L^G(z_1, \ldots, z_m)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq m$.

Theorem 2.2 ([1], Theorem 3.1)**.** *Let G be a finite group acting on $L(x)$, the rational function field of one variable over a field L. Assume that, for any*

$\sigma \in G, \sigma(L) \subset L$ and $\sigma(x) = a_\sigma x + b_\sigma$ for any $a_\sigma, b_\sigma \in L$ with $a_\sigma \neq 0$. Then $L(x)^G = L^G(z)$ for some $z \in L[x]$.

Theorem 2.3. *[(Kuniyoshi [2], Theorem 1.7] If char$K = p > 0$ and G is a finite p-group, then $K(G)$ is rational over K.*

The following Lemma can be extracted from some proofs in [5,6].

Lemma 2.4. *Let $\langle\tau\rangle$ be a cyclic group of order $n > 1$, acting on $K(v_1, \dots, v_{n-1})$, the rational function field of $n-1$ variables over a field K such that*

$$\tau \quad : \quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{n-1} \mapsto (v_1 \cdots v_{n-1})^{-1} \mapsto v_1.$$

If K contains a primitive n-th root of unity ξ, then $K(v_1, \dots, v_{n-1}) = K(s_1, \dots, s_{n-1})$ where $\tau : s_i \mapsto \xi^i s_i$ for $1 \leq i \leq n-1$.

3. Proof of Theorem 1.2. The case (i) follows from Theorem 2.3.

(ii) Let G be generated by an abelian normal subgroup H and an element α such that $\alpha^{p^a} \in H$. Assume that $H = \langle \alpha_1, \dots, \alpha_s \mid \alpha_i^{p^{a_i}} = 1, 1 \leq i \leq s\rangle$. We divide the proof into several steps. We are going now to find a faithful representation of G.

Step 1. Let V be a K-vector space whose dual space V^* is defined as $V^* = \bigoplus_{g\in G} K \cdot x(g)$, where G acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in G$. Thus $K(V)^G = K(x(g) : g \in G)^G = K(G)$.

Define $X_1, X_2, \dots, X_s \in V^*$ by

$$X_j = \sum_{\ell_1,\dots,\ell_s} x\left(\prod_{i\neq j} \alpha_i^{\ell_i}\right), \quad \text{for } 1 \leq j \leq s.$$

Note that $\alpha_i \cdot X_j = X_j$ for $j \neq i$. Let $\zeta_{p^{a_i}} \in K$ be a primitive p^{a_i}-th root of unity for $1 \leq i \leq s$. Define $Y_1, Y_2, \dots, Y_s \in V^*$ by

$$Y_i = \sum_{m=0}^{p^{a_i}-1} \zeta_{p^{a_i}}^{-m} \alpha_i^m \cdot X_i$$

for $1 \leq i \leq s$.

It follows that

$$\alpha_i \ : Y_i \mapsto \zeta_{p^{a_i}} Y_i, \ Y_j \mapsto Y_j, \text{ for } j \neq i.$$

Thus $\bigoplus_{1\leq j\leq s} K\cdot Y_j$ is a faithful representation space of the subgroup $H = \langle \alpha_1, \dots, \alpha_s\rangle$. The induced subspase W depends on the relations in G.

Define $x_{ji} = \alpha^i \cdot Y_j$ for $1 \leq j \leq s, 0 \leq i \leq p^a - 1$. Recall that G is of nilpotency class 2, so $[H, \alpha] \leq Z(G)$, i.e., $[\alpha_j, \alpha] = \gamma_j \in Z(G)$ for $1 \leq j \leq s$. It is not hard to see that

$$\alpha^{-i} \alpha_j \alpha^i = \alpha_j \gamma_j^i, \quad \text{for } 1 \leq j \leq s, 0 \leq i \leq p^a - 1.$$

We can now write the decomposition of γ_j in H:

$$(3.1)\ \ \gamma_j=\prod_{i=1}^{s}\alpha_i^{\alpha_{ij}p^{r_{ij}}}\quad \text{for } 1\le j\le s, r_{ij}\ge 0, \alpha_{ij}\in\mathbb{Z},\ \text{if}\ \alpha_{ij}\ne 0, \gcd(\alpha_{ij},p)=1.$$

Since $\alpha^{p^a}\in H$, we can write $\alpha^{p^a}=\alpha_1^{\delta_1}\cdots\alpha_s^{\delta_s}$ for some integers $\delta_1,\ldots,\delta_s$. It follows that

$$\begin{aligned}\alpha_j\ &: x_{ji}\mapsto \zeta_{p^{a_j}}\zeta_{p^{a_j}}^{i\alpha_{jj}p^{r_{jj}}}x_{ji},\ x_{mi}\mapsto\zeta_{p^{a_m}}^{i\alpha_{mj}p^{r_{mj}}}x_{mi},\ \text{for } m\ne j,\\ \alpha\ &: x_{j0}\mapsto x_{j1}\mapsto\cdots\mapsto x_{jp^a-1}\mapsto\zeta_{p^{a_j}}^{\delta_j}x_{j0},\end{aligned}$$

where $0\le i\le p^a-1$ and $1\le j\le s$. We find that $Y=\bigoplus_{j,i}K\cdot x_{ji}$ is a faithful G-subspace of V^*. Thus, by Theorem 2.1, it suffices to show that $K(x_{ji}:1\le j\le s, 0\le i\le p^a-1)^G$ is rational over K.

Step 2. In this step we will show that the case $\alpha^{p^a}\in H$ can be reduced to the case $\alpha^{p^a}=1$. Define $X_{ji}=x_{ji}/x_{ji-1}$ for $1\le j\le s$ and $1\le i\le p^a-1$. We have $K(x_{ji}:1\le j\le s,0\le i\le p^a-1)=K(x_{j0},X_{ji}:1\le j\le s,1\le i\le p^a-1)$ and $\alpha(x_{j0})=x_{j0}X_{j1}$ for all j. By applying Theorem 2.2 s times regarding x_{j0}'s, we get $K(x_{ji}:1\le j\le s,0\le i\le p^a-1)^G=K(X_{ji}:1\le j\le s,1\le i\le p^a-1)^G(U_1,\ldots,U_s)$, where $U_1,\ldots,U_s$ are fixed by the action of G. The action of G on X_{ji} is given by

$$\begin{aligned}\alpha_j\ &: X_{ji}\mapsto\zeta_{p^{a_j}}^{\alpha_{jj}p^{r_{jj}}}X_{ji},\ X_{mi}\mapsto\zeta_{p^{a_m}}^{\alpha_{mj}p^{r_{mj}}}X_{mi},\ \text{for } m\ne j,\\ \alpha\ &: X_{j1}\mapsto X_{j2}\mapsto\cdots\mapsto X_{jp^a-1}\mapsto\zeta_{p^{a_j}}^{\delta_j}(X_{j1}\cdots X_{jp^a-1})^{-1},\end{aligned}$$

where $1\le i\le p^a-1$ and $1\le j\le s$. Since K contains a primitive p^e-th root of unity ζ_{p^e}, where p^e is the exponent of G, from $\alpha^{p^a}=\alpha_1^{\delta_1}\cdots\alpha_s^{\delta_s}$ it follows that K contains as well the roots of unity $\zeta_{p^{a_j+a}}^{\delta_j}$. Define $Y_{ji}=X_{ji}/\zeta_{p^{a_j+a}}^{\delta_j}$ for $1\le i\le p^a-1$ and $1\le j\le s$. It follows that $K(X_{ji}:1\le j\le s,1\le i\le p^a-1)=K(Y_{ji}:1\le j\le s,1\le i\le p^a-1)$ and

$$\begin{aligned}\alpha_j\ &: Y_{ji}\mapsto\zeta_{p^{a_j}}^{\alpha_{jj}p^{r_{jj}}}Y_{ji},\ Y_{mi}\mapsto\zeta_{p^{a_m}}^{\alpha_{mj}p^{r_{mj}}}Y_{mi},\ \text{for } m\ne j,\\ \alpha\ &: Y_{j1}\mapsto Y_{j2}\mapsto\cdots\mapsto Y_{jp^a-1}\mapsto(Y_{j1}\cdots Y_{jp^a-1})^{-1}.\end{aligned}$$

Note that the parameters $\delta_1,\ldots,\delta_s$ of these groups G disappear in the above action. In conclusion, for any group G of nilpotency class 2, $K(G)$ is rational over $K(Y_{ji}:1\le j\le s,1\le i\le p^a-1)^G$. Thus all these fields $K(G)$ are isomorphic, and henceforth we will assume that $\alpha^{p^a}=1$.

Step 3. In this step we will find somewhat simpler actions of H. First, we can choose among all roots of unity $\zeta_{p^{a_m}}^{\alpha_{mj}p^{r_{mj}}}$, that appear in the actions of α_j's, the one that has the biggest order. Then we can change the indexing of α_j's and x_{mi}'s (regarding m), so that $\zeta_{p^{a_1}}^{\alpha_{11}p^{r_{11}}}$ has the biggest order. Therefore, for any m, j there exists an integer β_{mj} such that $\zeta_{p^{a_1}}^{\alpha_{11}p^{r_{11}}\beta_{mj}} = \zeta_{p^{a_m}}^{\alpha_{mj}p^{r_{mj}}}$. Under that renumbering, the actions of α_j's might undergo a slight change, that is

$$\alpha_j \ : x_{mi} \mapsto \zeta_{p^{a_m}}^{\epsilon_{mj}} \zeta_{p^{a_m}}^{i\alpha_{mj}p^{r_{mj}}} x_{mi}, \text{ for } 1 \leq m \leq s,$$

where $0 \leq i \leq p^a - 1, 1 \leq j \leq s; \epsilon_{m_j j} = 1$ for some $m_j : 1 \leq m_j \leq s; \epsilon_{nj} = 0$ for $n \neq m_j$ and $\epsilon_{m_j k} = 0$ for $k \neq j$.

Next, we can replace the generators of H with the new generators α_1, $\alpha_1^{-\beta_{12}}\alpha_2, \ldots, \alpha_1^{-\beta_{1s}}\alpha_s$. The actions of the new generators are of the following type

$$\alpha_1^{-\beta_{1j}}\alpha_j \ : x_{1i} \mapsto \zeta_{p^{b_{1j}}} x_{1i}, \ x_{mi} \mapsto \zeta_{p^{a_{mj}}} \zeta_{p^{b_{mj}}}^{i} x_{mi}, \text{ for } m \neq 1,$$

for some $b_{tj} \geq 0, a_{nj} \geq 0$, where $1 \leq t \leq s, 2 \leq j \leq s, 2 \leq n \leq s$ and $0 \leq i \leq p^a - 1$. Now, for $2 \leq m \leq s$ define $y_{mi} = x_{mi}x_{1i}^{-\beta_{m1}}$. Then we have

$$\alpha_1 \ : x_{1i} \mapsto \zeta_{p^{a_1}}^{\epsilon_{11}} \zeta_{p^{a_1}}^{i\alpha_{11}p^{r_{11}}} x_{1i}, \ y_{mi} \mapsto \zeta_{p^{c_{m1}}} y_{mi}, \text{ for } m \neq 1,$$
$$\alpha_1^{-\beta_{1j}}\alpha_j \ : x_{1i} \mapsto \zeta_{p^{b_{1j}}} x_{1i}, \ y_{mi} \mapsto \zeta_{p^{d_{mj}}} \zeta_{p^{c_{mj}}}^{i} y_{mi}, \text{ for } m \neq 1,$$

for some $c_{tj} \geq 0$ and $d_{tj} \geq 0$ $(2 \leq t, j \leq s)$, where $0 \leq i \leq p^a - 1$. Note that we obtained an action of α_1 such that i-th powers of roots of unity participate only in the action on x_{1i}, and also i-th powers of roots of unity do not appear in the actions of $\alpha_1^{-\beta_{12}}\alpha_2, \ldots, \alpha_1^{-\beta_{1s}}\alpha_s$ on x_{1i}.

Step 4. We can apply the same method of Step 3 with the remaining generators $\alpha_1^{-\beta_{12}}\alpha_2, \ldots, \alpha_1^{-\beta_{1s}}\alpha_s$. This process resembles the algorithm for matrix diagonalization by applying elementary transformations. In this way, we can eliminate all i-th powers of roots of unity, except on the diagonal actions. Thus we will finally obtain generators $\beta_1, \ldots, \beta_s$ of H which act on the function field $K(y_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1) = K(x_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1)$ in this way

$$\beta_j \ : y_{ji} \mapsto \zeta_{p^{b_j}} \zeta_{p^{b_{jj}}}^{i} y_{ji}, \ y_{mi} \mapsto \zeta_{p^{b_{mj}}} y_{mi}, \text{ for } m \neq j,$$

for some $b_j \geq 0, b_{tj} \geq 0$, where $1 \leq t, j \leq s$ and $0 \leq i \leq p^a - 1$. Clearly, the action of α is not changed:

$$\alpha \ : y_{j0} \mapsto y_{j1} \mapsto \cdots \mapsto y_{jp^a-1} \mapsto y_{j0},$$

where $1 \leq j \leq s$.

Observe that from $1 = [\alpha_j, \alpha^{p^a}] = \gamma_j^{p^a}$ it follows that $a_m - r_{mj} \leq a$ for all m, j (see (3.1)). Since each primitive root $\zeta_{p^{b_{jj}}}$ is obtained via multiplication of roots of degree $p^{a_m - r_{mj}}$, we deduce that $b_{jj} \leq a$ for all j.

Clearly, $W = \bigoplus_{j,i} K \cdot y_{ij} \subset V^*$ is the induced G-subspace obtained from V. Thus, by Theorem 2.1 it suffices to show that W^G is rational over K.

Step 5. If we suppose that $b_{jj} = 0$ for all $1 \le j \le s$, this would mean that G is abelian, i.e., of nilpotency class 1. Therefore, we may assume that $b_{jj} \ge 1$, where $1 \le j \le t$ for some $t : 1 \le t \le s$, and if $t < s$, $b_{jj} = 0$ for all $t+1 \le j \le s$. (Here we can achieve this by renumbering β_j's.) Denote $A = b_{11}$ and $\xi = \zeta_{p^{a-A}}$, a primitive p^{a-A}-th root of unity. For $0 \le \ell \le p^{a-A}-1$ and $0 \le k \le p^A - 1$ define

$$u_{\ell k} = y_{1k} + \xi^\ell y_{1,k+p^A} + (\xi^\ell)^2 y_{1,k+2p^A} + \cdots + (\xi^\ell)^{p^{a-A}-1} y_{1,k+(p^{a-A}-1)p^A}.$$

It is not hard to see that this is a well defined non-singular transformation, so $K(u_{\ell k} : 0 \le \ell \le p^{a-A}-1, 0 \le k \le p^A-1) = K(y_{1i} : 0 \le i \le p^a - 1)$. Then the actions of the generators of G on $K(u_{\ell k})$ are

$$\begin{aligned} \beta_1 &: u_{\ell k} \mapsto \zeta_{p^{b_1}} \zeta_{p^A}^k u_{\ell k}, \\ \beta_m &: u_{\ell k} \mapsto \zeta_{p^{b_{1m}}} u_{\ell k}, \qquad \text{for } m \ne 1, \\ \alpha &: u_{\ell 0} \mapsto u_{\ell 1} \mapsto \cdots \mapsto u_{\ell p^A-1} \mapsto \xi^{-\ell} u_{\ell 0}, \end{aligned}$$

where $0 \le \ell \le p^{a-A}-1$ and $0 \le k \le p^A - 1$.

For $1 \le i \le p^A - 1$ define $v_{0i} = u_{0i}/u_{0i-1}$. For $1 \le \ell \le p^{a-A}-1$ and $0 \le k \le p^A - 1$ define $v_{\ell k} = u_{\ell k}/u_{0k}$. Thus $K(u_{\ell k} : 0 \le \ell \le p^{a-A}-1, 0 \le k \le p^A - 1) = K(u_{00}, v_{\ell k} : (k,\ell) \ne (0,0))$, and for every $g \in G$

$$g \cdot u_{00} \in K(v_{\ell k} : (k,\ell) \ne (0,0)) \cdot u_{00}, \tag{3.2}$$

while the subfield $K(v_{\ell k} : (k,\ell) \ne (0,0))$ is invariant by the action of G, i.e.,

$$\begin{aligned} \beta_1 &: v_{0i} \mapsto \zeta_{p^A} v_{0i},\ v_{\ell k} \mapsto v_{\ell k}, \qquad \text{for } 1 \le i \le p^A-1, 0 \le k \le p^A - 1, \ell \ne 0 \\ \beta_m &: v_{\ell k} \mapsto v_{\ell k}, \qquad \text{for } m \ne 1, 0 \le \ell \le p^{a-A}-1, 0 \le k \le p^A-1, (k,\ell) \ne (0,0), \\ \alpha &: v_{01} \mapsto v_{02} \mapsto \cdots \mapsto v_{0p^A-1} \mapsto (v_{01} \cdots v_{0p^A-1})^{-1}, \\ &\quad v_{\ell 0} \mapsto v_{\ell 1} \mapsto \cdots \mapsto v_{\ell p^A-1} \mapsto \xi^{-\ell} v_{\ell 0}, \qquad \text{for } 1 \le \ell \le p^{a-A}-1. \end{aligned}$$

Note that we can apply Theorem 2.2 on the variable u_{00} so that we may assume that u_{00} is invariant under the action of G.

Step 6. We can apply the same type of transformations as in Step 5 for each $K(y_{ji} : 0 \le i \le p^a - 1)$, where $2 \le j \le t$. For $t+1 \le j \le s$ define $w_{ji} = y_{ji}/y_{ji-1}$, where $1 \le i \le p^a - 1$. Denote by $\mathcal{X}$ the set of all new variables (i.e., of type $v_{\ell k}$ from Step 5, applied for $1 \le j \le t$, w_{ji}, and the variables obtained by applying Theorem 2.2 on the variables of type (3.2)) and by $\mathcal{X}_j$ the subset of all variables that are not invariant under β_j and are not of the type (3.2). Observe that β_1 leaves invariant all new variables except v_{0i}'s and except the type (3.2), i.e. $\mathcal{X}_1 = \{v_{0i} : 1 \le i \le p^A - 1\}$.

Define $w_1 = v_{01}^{p^A}$ and $w_i = v_{0i}/v_{0i-1}$ for $2 \le i \le p^A - 1$. Then $K(y_{ji} : 0 \le i \le p^a - 1, 1 \le j \le s)^{\langle \beta_1 \rangle} = K(\mathcal{X})^{\langle \beta_1 \rangle} = K(w_i, \mathcal{X} \setminus \mathcal{X}_1, 1 \le i \le p^A - 1)$. The action of α on $\mathcal{X}_1$ is

$$\begin{aligned} \alpha \ : & w_1 \mapsto w_1 w_2^{p^A}, \\ & w_2 \mapsto w_3 \mapsto \cdots \mapsto w_{p^A-1} \mapsto (w_1 w_2^{p^A-1} w_3^{p^A-2} \cdots w_{p^A-1}^2)^{-1} \mapsto \\ & \mapsto w_1 w_2^{p^A-2} w_3^{p^A-3} \cdots w_{p^A-2}^2 w_{p^A-1} \mapsto w_2. \end{aligned}$$

Define $z_1 = w_2, z_i = \alpha^i \cdot w_2$ for $2 \le i \le p^A - 1$. Now the action of α is

$$\alpha \ : z_1 \mapsto z_2 \mapsto \cdots \mapsto z_{p^A-1} \mapsto (z_1 z_2 \cdots z_{p^A-1})^{-1}.$$

Since $w_1 = (z_{p^A-1} z_1^{p^A-1} z_2^{p^A-2} \cdots z_{p^A-2}^2)^{-1}$, we get that $K(w_1, \ldots, w_{p^A-1}) = K(z_1, \ldots, z_{p^A-1})$. From Lemma 2.4 it follows that the action of α on $K(z_1, \ldots, z_{p^A-1})$ can be linearized. Clearly, the action of α on $K(v_{\ell i} : 1 \le \ell \le p^{a-A} - 1, 0 \le i \le p^A - 1)$ is also linear.

Similarly, we can linearize the action of α on $K(\mathcal{X})^{\langle \beta_j \rangle}$ for each $j : 2 \le j \le t$. For $j : t + 1 \le j \le s$ the action of α on $K(w_{ji})$ is

$$\alpha \ : w_{j1} \mapsto w_{j2} \mapsto \cdots \mapsto w_{jp^a-1} \mapsto (w_{j1} w_{j2} \cdots w_{jp^a-1})^{-1},$$

which again can be linearized according to Lemma 2.4. Therefore, we obtain a linear action of α on $K(\mathcal{X})^H$. We are done.

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