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A STUDY ON LACUNARY STRONG CONVERGENCE ACCORDING TO MODULUS FUNCTIONS

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Abstract

In this article, we study a new generalization of the lacunary strongly convergent sequences and introduce the concept of lacunary strong convergence according to g^k for sequences of complex (or real) numbers, where $g^k = g \circ g \circ \cdots \circ g$ (k times) represents a composite modulus function. After that, we determine the connections of lacunary strong convergence and lacunary statistical convergence to lacunary strong convergence according to g^k. Furthermore, we investigate several properties of this generalization.

Key words: sequence space, modulus function, lacunary sequence, strong convergence, statistical convergence

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1. Introduction. In this article, the spaces of all bounded and convergent sequences, respectively, are indicated by the symbols ℓ_∞ and c as well as the sets of all complex and natural numbers, respectively, are indicated by the symbols $\mathbb{C}$ and $\mathbb{N}$.

The concept of the modulus function has been introduced for the first time by Nakano [1] in 1953. For this, we remember that g is indeed a modulus function from $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{R}^+ \cup \{0\}$ such that the next properties hold:

(i) $g(c) = 0$ if and only if $c = 0$,

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(ii) $g(c_1 + c_2) \leq g(c_1) + g(c_2)$ for every $c_1, c_2 \in \mathbb{R}^+ \cup \{0\}$,

(iii) g is increasing,

(iv) g is continuous from the right at 0.

Since $|g(c_1) - g(c_2)| \leq g(|c_1 - c_2|)$, then from property (iv), it is clear that g is continuous on $\mathbb{R}^+ \cup \{0\}$. It is conceivable for a modulus to either be bounded or not. For illustration, the function $g(c) = \frac{c}{c+1}$ is a bounded modulus, but $h(c) = \sqrt{c}$ is being an unbounded modulus function. Additionally, from the second property of the modulus functions, we have $g(mc) \leq mg(c)$ for every $m \in \mathbb{N}$.

Çolak [2], Bhardwaj and Dhawan [3], Altin et al. [4], Mohiuddine et al. [5], Candan [6], Bektaş and Atici [7] and many other authors have built and discussed certain sequence spaces with the help of modulus functions.

The natural density $\delta(E)$ of a subset E in $\mathbb{N}$ is defined by

$$\delta(E) = \lim_{k\to\infty} \frac{1}{k}|E_k|,$$

where $|E_k| = |\{i \leq k \colon i \in E\}|$ indicates the number of elements of E not more than k. It is clear that $\delta(\mathbb{N}) = 1$ and $\delta(E) = 0$ if E is a finite subset in $\mathbb{N}$ and $\delta(\mathbb{N}\setminus E) = 1 - \delta(E)$.

From Freedman et al. [8], the sequence $\theta = (k_r)$ of non-negative integer numbers is implied as a lacunary sequence such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. And the periods provided by θ are indicated by $I_r = (k_{r-1}, k_r]$ and $\frac{k_r}{k_{r-1}}$ is shortened by q_r. These notations shall be implemented throughout the article.

Lacunary sequences were studied by many authors in [4–7,9–17].

The following expression is the definition that Fridy and Orhan [18] meant from lacunary statistical convergence.

Suppose that $\theta = (k_r)$ is a given lacunary sequence. A sequence (ξ_i) of numbers is defined to be lacunary statistically convergent (or S_θ-convergent) to ξ, if

$$\lim_{r\to\infty} \frac{1}{h_r}|\{i \in I_r \colon |\xi_i - \xi| \geq \varepsilon\}| = 0$$

for each given $\varepsilon > 0$. In this manner, we shall write $\xi_i \to \xi(S_\theta)$ or $S_\theta - \lim \xi_i = \xi$. From the entire paper, the class of S_θ-convergent sequences is indicated by S_θ, that is

$$S_\theta = \left\{(\xi_i) \colon \lim_{r\to\infty} \frac{1}{h_r}|\{i \in I_r \colon |\xi_i - \xi| \geq \varepsilon\}| = 0 \text{ for some } \xi \in \mathbb{C}\right\}.$$

Lacunary statistical convergence has also been talked in [10,12,14,19,20] and studied by many other authors.

According to Freedman et al. [8], the space N_θ of lacunary strongly convergent sequences has been introduced as follows:

$$N_\theta = \left\{ (\xi_i)\colon \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} |\xi_i - \xi| = 0 \text{ for some } \xi \right\}.$$

This space is indeed a BK-space with the norm defined below

$$\|\xi\|_\theta = \sup_r \frac{1}{h_r} \sum_{i\in I_r} |\xi_i|.$$

The set of sequences in which $\xi = 0$ in the definition of N_θ is indicated by N_θ^0. It is also a BK-space with the given norm $\|\cdot\|_\theta$. After that, by the use of a modulus function, the concept of lacunary strong convergence was generalized by Pehlivan and Fisher [14].

Definition 1.1 ([14])**.** Suppose that $\theta = (k_r)$ is a given lacunary sequence, and let g be a modulus function. Then the space of lacunary strongly convergent sequences with respect to f is indicated by $N_\theta(g)$ and defined as follows:

$$N_\theta(g) = \{(\xi_i)\colon \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} g(|\xi_i - \xi|) = 0 \text{ for some } \xi\}.$$

In the case of $g(c) = c$, the space $N_\theta(g)$ will become the space N_θ.

Lemma 1.1 ([4])**.** *If g is a modulus function, then for each $k \in \mathbb{N}$ the function $g^k = g \circ g \circ \cdots \circ g$ (k times) also becomes a modulus.*

This notation is implemented throughout the next sections.

2. Main results. In this part of our study, we shall prove some results on the sequence space $N_\theta(g^k)$ which involves determining the relations between N_θ and $N_\theta(g^k)$, $N_\theta(g^k)$ and $N_\theta(g^n)$, $N_{\theta'}(g^k)$ and $N_\theta(g^n)$, S_θ and $N_\theta(g^k)$, where g represents a modulus function under certain properties and $k, n \in \mathbb{N}$.

Definition 2.1. Suppose that $\theta = (k_r)$ is a given lacunary sequence and let g be a modulus function. Then the sequence (ξ_i) in $\mathbb{C}$ is defined to be lacunary strongly convergent according to g^k (or $N_\theta(g^k)$-strongly convergent) to some $\xi \in \mathbb{C}$, if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} g^k(|\xi_i - \xi|) = 0.$$

In this manner, we shall write $\xi_i \to \xi(N_\theta(g^k))$ or $N_\theta(g^k) - \lim \xi_i = \xi$. The collection of all $N_\theta(g^k)$-strongly convergent sequences is indicated by $N_\theta(g^k)$. That is

$$N_\theta(g^k) = \{(\xi_i)\colon \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} g^k(|\xi_i - \xi|) = 0 \text{ for some } \xi \in \mathbb{C}\}.$$

Notice that in this definition the function g is not required to be an unbounded modulus. If we put $g(c) = c$, then the $N_\theta(g^k)$-strong convergence is reduced to the N_θ-strong convergence. $N_\theta^0(g^k)$ is indicated as the collection of all sequences in which $\xi = 0$ in the definition of $N_\theta(g^k)$. It is clear that if $k = 1$, then the spaces $N_\theta(g^k)$ and $N_\theta^0(g^k)$ will, respectively, become the same as $N_\theta(g)$ and $N_\theta^0(g)$ of Pehlivan and Fisher [14]. In the case of $\theta = (2^r)$ and $g(c) = c$, we see that $N_\theta(g^k) = |\sigma_1|$, where $|\sigma_1|$ is the set of strongly Cesaro summable sequences [2] as defined by

$$|\sigma_1| = \left\{ (\xi_i) \colon \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\xi_i - \xi| = 0 \text{ for some } \xi \right\}.$$

Theorem 2.1. *Consider a lacunary sequence $\theta = (k_r)$ and let g be a modulus function. Then the sets $N_\theta(g^k)$ and $N_\theta^0(g^k)$ are linear spaces.*

Proof. We here only consider $N_\theta(g^k)$. Let (ξ_i), $(\zeta_i) \in N_\theta(g^k)$ and $\alpha, \beta \in \mathbb{C}$. Then there will be natural numbers A_α and B_β such that $|\alpha| \le A_\alpha$ and $|\beta| \le B_\beta$. According to the definition of g, we have

$$\begin{aligned}\frac{1}{h_r} \sum_{i\in I_r} g^k(|(\alpha\xi_i + \beta\zeta_i) - (\alpha\xi + \beta\zeta)|) &= \frac{1}{h_r} \sum_{i\in I_r} g^k(|\alpha(\xi_i - \xi) + \beta(\zeta_i - \zeta)|) \\ &\le A_\alpha \frac{1}{h_r} \sum_{i\in I_r} g^k(|\xi_i - \xi|) + B_\beta \frac{1}{h_r} \sum_{i\in I_r} g^k(|\zeta_i - \zeta|).\end{aligned}$$

Then adding the limits on both given sides as r tends to ∞, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} g^k(|(\alpha\xi_i + \beta\zeta_i) - (\alpha\xi + \beta\zeta)|) = 0.$$

Therefore $N_\theta(g^k)$ is indeed a linear space. Hence the proof. □

Theorem 2.2. *Consider a lacunary sequence $\theta = (k_r)$ and let g be a modulus function. If $g(c) \le c$ for every $c \in \mathbb{R}^+ \cup \{0\}$, then every lacunary strongly convergent sequence implies $N_\theta(g^k)$-strongly convergent, i.e., $N_\theta \subset N_\theta(g^k)$.*

Proof. Since g is increasing and $g(c) \le c$, then

$$g^k(c) \le g^{k-1}(c) \le \cdots \le g(c) \le c. \tag{2.1}$$

Now if $(\xi_i) \in N_\theta$, the proof follows from

$$\frac{1}{h_r} \sum_{i\in I_r} g^k(|\xi_i - \xi|) \le \frac{1}{h_r} \sum_{i\in I_r} g^{k-1}(|\xi_i - \xi|) \le \cdots \le \frac{1}{h_r} \sum_{i\in I_r} g(|\xi_i - \xi|) \le \frac{1}{h_r} \sum_{i\in I_r} |\xi_i - \xi|.$$

That is,

$$\frac{1}{h_r} \sum_{i\in I_r} g^k(|\xi_i - \xi|) \le \frac{1}{h_r} \sum_{i\in I_r} |\xi_i - \xi|.$$

Then adding the limits on both given sides as r tends to ∞, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i-\xi|)=0.$$

Therefore, we obtain that $(\xi_i)\in N_\theta(g^k)$. Hence the proof. □

The instance, which is mentioned below, clarifies the strictness of the above inclusion relationship.

Suppose $\theta=(k_r)$ is a given lacunary sequence. Define (γ_i) as γ_i is assumed to be $[\sqrt{h_r}]$ for the first $[\sqrt{h_r}]$ integers in I_r, and $\gamma_i=0$ otherwise, where $[c]$ indicates an integral component of the real number c. Now let $(\xi_i)=(\gamma_i)$ and $g(c)=\frac{c}{c+1}$, then $g(c)\le c$ for every $c\in\mathbb{R}^+\cup\{0\}$ and $g^k(c)=\frac{c}{kc+1}$. For that, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i|)=\lim_{r\to\infty}\frac{1}{h_r}\frac{[\sqrt{h_r}][\sqrt{h_r}]}{(k[\sqrt{h_r}]+1)}=0.$$

But

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}|\xi_i|=\lim_{r\to\infty}\frac{1}{h_r}[\sqrt{h_r}][\sqrt{h_r}]=1.$$

Hence $(\xi_i)\in N_\theta(g^k)-N_\theta$.

Corollary 2.3. *Suppose that $\theta=(k_r)$ is a given lacunary sequence and let g be a modulus function. If $g(c)\le c$ for every $c\in\mathbb{R}^+\cup\{0\}$, then*

$$N_\theta\subset N_\theta(g)\subset\cdots\subset N_\theta(g^{k-1})\subset N_\theta(g^k).$$

Theorem 2.4. *Consider a lacunary sequence $\theta=(k_r)$ and let g be a modulus function. If $g(c)\ge c$ for every $c\in\mathbb{R}^+\cup\{0\}$, then every $N_\theta(g^k)$-strongly convergent sequence implies lacunary strongly convergent, i.e., $N_\theta(g^k)\subset N_\theta$.*

Proof. Since g is increasing and $g(c)\ge c$, then

$$g^k(c)\ge g^{k-1}(c)\ge\cdots\ge g(c)\ge c. \tag{2.2}$$

Now if $(\xi_i)\in N_\theta(g^k)$, the proof follows from

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i-\xi|)\ge\frac{1}{h_r}\sum_{i\in I_r} g^{k-1}(|\xi_i-\xi|)\ge\cdots\ge\frac{1}{h_r}\sum_{i\in I_r} g(|\xi_i-\xi|)\\ \ge\frac{1}{h_r}\sum_{i\in I_r}|\xi_i-\xi|.\end{aligned}$$

That is

$$\frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i-\xi|)\ge\frac{1}{h_r}\sum_{i\in I_r}|\xi_i-\xi|.$$

Then adding the limits on both given sides as r tends to ∞, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}|\xi_i-\xi|=0.$$

Therefore, we obtain that $(\xi_i)\in N_\theta$. Hence the proof. $\square$

Giving example for showing the strictness of the above inclusion relationship is omitted, so we leave it as an open problem.

Corollary 2.5. *Suppose that $\theta=(k_r)$ is a given lacunary sequence and let g be a modulus function. If $g(c)\geq c$ for every $c\in\mathbb{R}^+\cup\{0\}$, then*

$$N_\theta(g^k)\subset N_\theta(g^{k-1})\subset\cdots\subset N_\theta(g)\subset N_\theta.$$

Corollary 2.6. *Suppose that $\theta=(k_r)$ is a given lacunary sequence and let g be a modulus function with $k<n$.*

(i) *If $g(c)\leq c$ for every $c\in\mathbb{R}^+\cup\{0\}$, then $N_\theta(g^k)\subset N_\theta(g^n)$.*

(ii) *If $g(c)\geq c$ for every $c\in\mathbb{R}^+\cup\{0\}$, then $N_\theta(g^n)\subset N_\theta(g^k)$.*

The proof of (i) and (ii) is clearly done by using (2.1) and (2.2), respectively.

Theorem 2.7. *Consider a lacunary sequence $\theta=(k_r)$ and let g and h be two modulus functions. If $g(c)\leq h(c)$ for every $c\in\mathbb{R}^+\cup\{0\}$, then:*

(i) $N_\theta(g^k\circ h)\subset N_\theta(g^{k+1})$.

(ii) $N_\theta(h^{k+1})\subset N_\theta(h^k\circ g)$.

Proof. (i) Let $(\xi_i)\in N_\theta(g^k\circ h)$. Since $g(c)\leq h(c)$, then according to (2.1), we may have $g^{k+1}(c)\leq g^k(h(c))$ and so

$$\frac{1}{h_r}\sum_{i\in I_r}g^{k+1}(|\xi_i-\xi|)\leq\frac{1}{h_r}\sum_{i\in I_r}g^k(h(\lfloor\xi_i-\xi\rfloor)).$$

Then adding the limits on both given sides as r tends to ∞, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}g^{k+1}(|\xi_i-\xi|)=0.$$

So, we conclude that $(\xi_i)\in N_\theta(g^{k+1})$. The proof of (ii) follows from $h^k(g(c))\leq h^{k+1}(c)$. Hence the proof. $\square$

Providing examples to demonstrate the strictness of the aforementioned inclusion relationships are omitted, so we leave them as open problems.

Theorem 2.8. *Consider the lacunary sequences $\theta = (k_r)$ and $\theta' = (w_r)$ such that $I_r \subseteq I'_r$ for all $r \in \mathbb{N}$, and let g be a modulus function. If $\lim_{r\to\infty} \frac{\ell_r}{h_r} = 1$ and $(\xi_i) \in \ell_\infty$, then $N_\theta(g^k) \subset N_{\theta'}(g^k)$.*

Proof. Let $(\xi_i) \in \ell_\infty \cap N_\theta(g^k)$. Since (ξ_i) is a bounded sequence, then there is some $H > 0$ such that $|\xi_i - \xi| \le H$ for all $i \in \mathbb{N}$. Since $I_r \subseteq I'_r$ and $h_r \le \ell_r$ for all $r \in \mathbb{N}$, we shall write

$$\begin{aligned} \frac{1}{\ell_r} \sum_{i\in I'_r} g^k(|\xi_i - \xi|) &= \frac{1}{\ell_r} \sum_{i\in I'_r - I_r} g^k(|\xi_i - \xi|) + \frac{1}{\ell_r} \sum_{i\in I_r} g^k(|\xi_i - \xi|) \\ &\le \left(\frac{\ell_r - h_r}{\ell_r}\right) g^k(H) + \frac{1}{\ell_r} \sum_{i\in I_r} g^k(|\xi_i - \xi|) \\ &\le \left(\frac{\ell_r}{h_r} - 1\right) g^k(H) + \frac{1}{h_r} \sum_{i\in I_r} g^k(|\xi_i - \xi|). \end{aligned}$$

Then adding the limits on both given sides as r tends to ∞, we have

$$\lim_{r\to\infty} \frac{1}{\ell_r} \sum_{i\in I'_r} g^k(|\xi_i - \xi|) = 0,$$

since $\lim_{r\to\infty} \frac{\ell_r}{h_r} = 1$. Therefore, we obtain that $(\xi_i) \in N_{\theta'}(g^k)$. Hence the proof. □

Corollary 2.9. *Suppose that g is a modulus function and $k < n$. If $g(c) \le c$ for every $c \in \mathbb{R}^+ \cup \{0\}$, then from Theorem 2.8 we obtain:*

(i) $N_\theta \subset N_{\theta'}(g^k)$.

(ii) $N_\theta(g^k) \subset N_{\theta'}(g^n)$.

Corollary 2.10. *Suppose that g is a modulus function and $k < n$. If $g(c) \ge c$ for every $c \in \mathbb{R}^+ \cup \{0\}$, then from Theorem 2.8 we obtain:*

(i) $N_\theta(g^k) \subset N_{\theta'}$.

(ii) $N_\theta(g^n) \subset N_{\theta'}(g^k)$.

Theorem 2.11. *Consider the lacunary sequences $\theta = (k_r)$ and $\theta' = (w_r)$ such that $I_r \subseteq I'_r$ for all $r \in \mathbb{N}$, and let g be a modulus function. If $\limsup_{r\to\infty} \frac{\ell_r}{h_r} < \infty$, then $N_{\theta'}(g^k) \subset N_\theta(g^k)$.*

Proof. Assume that g is a modulus function and $(\xi_i) \in N_{\theta'}(g^k)$. Since $I_r \subseteq I'_r$ we shall write

$$\frac{1}{\ell_r} \sum_{i\in I_r} g^k(|\xi_i - \xi|) \le \frac{1}{\ell_r} \sum_{i\in I'_r} g^k(|\xi_i - \xi|)$$

and so

$$\frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i-\xi|) \le \frac{\ell_r}{h_r}\frac{1}{\ell_r}\sum_{i\in I'_r} g^k(|\xi_i-\xi|).$$

Then adding the limits on both given sides as r tends to ∞, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i-\xi|)=0.$$

Therefore, we obtain that $(\xi_i)\in N_\theta(g^k)$. Hence the proof. □

Corollary 2.12. *Suppose that g is a modulus function and $k<n$. If $g(c)\le c$ for every $c\in\mathbb{R}^+\cup\{0\}$, then from Theorem 2.11, we obtain:*

(i) $N_{\theta'}\subset N_\theta(g^k)$.

(ii) $N_{\theta'}(g^k)\subset N_\theta(g^n)$.

Corollary 2.13. *Suppose that g is a modulus function and $k<n$. If $g(c)\ge c$ for every $c\in\mathbb{R}^+\cup\{0\}$, then from Theorem 2.11, we obtain:*

(i) $N_{\theta'}(g^k)\subset N_\theta$.

(ii) $N_{\theta'}(g^n)\subset N_\theta(g^k)$.

Theorem 2.14. *Consider a lacunary sequence $\theta=(k_r)$ and let g be a modulus function. If a sequence is $N_\theta(g^k)$-strongly convergent to ξ, then it is S_θ-convergent to ξ, i.e., $N_\theta(g^k)\subset S_\theta$.*

Proof. Suppose that $(\xi_i)\in N_\theta(f^k)$ and $\varepsilon>0$ be given, then we may write

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i-\xi|) &= \frac{1}{h_r}\sum_{\substack{i\in I_r\\ |\xi_i-\xi|\ge\varepsilon}} g^k(|\xi_i-\xi|)+\frac{1}{h_r}\sum_{\substack{i\in I_r\\ |\xi_i-\xi|<\varepsilon}} g^k(|\xi_i-\xi|)\\ &\ge \frac{1}{h_r}\sum_{\substack{i\in I_r\\ |\xi_i-\xi|\ge\varepsilon}} g^k(|\xi_i-\xi|)\\ &\ge \frac{1}{h_r}|\{i\in I_r\colon |\xi_i-\xi|\ge\varepsilon\}|g^k(\varepsilon).\end{aligned}$$

Then adding the limits on both given sides as r tends to ∞, we obtain that $(\xi_i)\in S_\theta$. Hence the proof. □

Theorem 2.15. *Consider a lacunary sequence $\theta=(k_r)$ and let g be a modulus function. If a sequence is bounded and S_θ-convergent to ξ, then it is $N_\theta(g^k)$-strongly convergent to ξ, i.e., $\ell_\infty\cap S_\theta\subset N_\theta(g^k)$.*

Proof. Suppose that (ξ_i) is a bounded sequence and $S_\theta - \lim \xi_i = \xi$ with a given $\varepsilon > 0$. Then there is some $H > 0$ such that $|\xi_i - \xi| \leq H$ for every i. Now we may write

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i-\xi|) &= \frac{1}{h_r}\sum_{\substack{i\in I_r\\ |\xi_i-\xi|\geq\varepsilon}} g^k(|\xi_i-\xi|) + \frac{1}{h_r}\sum_{\substack{i\in I_r\\ |\xi_i-\xi|<\varepsilon}} g^k(|\xi_i-\xi|)\\ &\leq \frac{1}{h_r}\sum_{\substack{i\in I_r\\ |\xi_i-\xi|\geq\varepsilon}} g^k(H) + \frac{1}{h_r}\sum_{\substack{i\in I_r\\ |\xi_i-\xi|<\varepsilon}} g^k(\varepsilon)\\ &\leq g^k(H)\frac{1}{h_r}|\{i\in I_r\colon |\xi_i-\xi|\geq\varepsilon\}| + \frac{h_r}{h_r}g^k(\varepsilon).\end{aligned}$$

Then, by adding the limits on both given sides as r tends to ∞, we obtain that the sequence (ξ_i) is $N_\theta(g^k)$-strongly convergent to ξ. Hence the proof. □

Theorem 2.16. *Consider a lacunary sequence $\theta = (k_r)$ and let g be a modulus function. Then $N_\theta(g^k) - \lim \xi_i = \xi$ is unique.*

Proof. Suppose that $N_\theta(g^k) - \lim \xi_i = \xi_1$ and $N_\theta(g^k) - \lim \xi_i = \xi_2$. Then

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_1-\xi_2|) &= \frac{1}{h_r}\sum_{i\in I_r} g^k(|(\xi_i-\xi_1)-(\xi_i-\xi_2)|)\\ &\leq \frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i-\xi_1|+|\xi_i-\xi_2|)\\ &\leq \frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i-\xi_1|) + \frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_i-\xi_2|).\end{aligned}$$

Then adding the limits on both given sides as r tends to ∞, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} g^k(|\xi_1-\xi_2|) = 0.$$

So we obtain that $\xi_1 = \xi_2$. Hence the proof. □

3. Conclusions. In the view of this article, the concept of the lacunary strongly convergent sequence according to a composite modulus function has been introduced. After that, by the use of a modulus function g depending on certain properties, and lacunary sequences $\theta = (k_r)$ and $\theta' = (w_r)$ with positive integers k and n, the relations between the sets N_θ and $N_\theta(g^k)$, $N_\theta(g^k)$ and $N_\theta(g^n)$, $N_{\theta'}(g^k)$ and $N_\theta(g^n)$, S_θ and $N_\theta(g^k)$ have been determined.

In addition, there is a strong chance that further findings can be obtained from this research field. In fact, by selecting distinct modulus functions g and h and including positive integers k and n, many spaces are possibly obtained for sophisticated applications in relevant subjects.

REFERENCES

[1] Nakano H. (1953) Concave modulars, J. Math. Soc. Japan, **5**(1), 29–49, https://doi.org/10.2969/jmsj/00510029.

[2] Çolak R. (2003) Lacunary strong convergence of difference sequences with respect to a modulus function, Filomat, 9–14, http://www.jstor.org/stable/43998646.

[3] Bhardwaj V. K., S. Dhawan (2016) Density by moduli and lacunary statistical convergence, Abstract and Applied Analysis, Hindawi Publ. Corp., Art. ID 9365037, 11 pp, https://doi.org/10.1155/2016/9365037.

[4] Altin Y., H. Altinok, R. Çolak (2006) On some seminormed sequence spaces defined by a modulus function, Kragujevac J. Math., **29**(29), 121–132.

[5] Mohiuddine S. A., K. Raj, A. Choudhary (2022) Difference sequence spaces based on Lucas band matrix and modulus function, São Paulo J. Math. Sci., 1–12. https://doi.org/10.1007/s40863-020-00203-2.

[6] Candan M. (2013) Vector-valued FK-spaces defined by a modulus function and an infinite matrix, Thai J. Math., **12**(1), 155–165.

[7] Bektaş Ç. A., G. Atici (2011) Some new type of lacunary generalized difference sequence spaces defined by a sequence of moduli, Int. J. Open Probl. Compt. Math, **4**(2), http://www.ijopcm.org/Vol/11/IJOPCM(vol.4.2.16.J.11).pdf.

[8] Freedman A. R., J. J. Sember, M. Raphael (1978) Some Cesaro-type summability spaces, Proc. London Math. Soc., **3**(3), 508–520. https://doi.org/10.1112/plms/s3-37.3.508.

[9] Cakalli H., C. G. Aras, A. Sonmez (2015) Lacunary statistical ward continuity, in: AIP Conf. Proc., AIP Publ. LLC, **1676**(1), 020042(1–6), https://doi.org/10.1063/1.4930468.

[10] Ibrahim I. S., R. Çolak (2021) On strong lacunary summability of order α with respect to modulus functions, Ann. Univ. Craiova – Math. Comp. Sci. Ser., **48**(1), 127–136, https://doi.org/10.52846/ami.v48i1.1399.

[11] Işik M., K. E. Et (2015) On lacunary statistical convergence of order α in probability, in: AIP Conf. Proc., AIP Publ. LLC, **1676**(1), 020045(1–3), https://doi.org/10.1063/1.4930471.

[12] Mursaleen M., K. Raj, S. K. Sharma (2015) Some spaces of difference sequences and lacunary statistical convergence in n-normed space defined by a sequence of Orlicz functions, Miskolc Math. Notes, **16**(1), 283–304, https://doi.org/10.18514/MMN.2015.754.

[13] Kaplan H., H. Cakalli (2016) Variations on strong lacunary quasi-Cauchy sequences, J. Nonlinear Sci. Appl, **9**(6), 4371–4380, https://doi.org/10.22436/jnsa.009.06.77.

[14] Pehlivan S., B. Fisher (1994) On some sequence spaces, Ind. J. Pure Appl. Math., **25**, 1067–1072.

[15] Şengül H. (2017) Some Cesàro-type summability spaces defined by a modulus function of order $(\alpha;\beta)$, Comm. Fac. Sci. Univ. Ankara, Ser. A1 Math. Stat., **66**(2), 80–90, https://doi.org/10.1501/Commua1_0000000803.

[16] Yildiz Ş. (2017) Lacunary statistical delta 2 quasi Cauchy sequences, Sakarya Univ. J. Sci., **21**(6), 1408–1412, https://doi.org/10.16984/saufenbilder.336128.

[17] Aral N. D., H. S. Kandemir, M. Et (2023) Strongly lacunary convergence of order β of difference sequences of fractional order in neutrosophic normed spaces, FILOMAT, **37**(19), 6443–6451.
[18] Fridy J. A., C. Orhan (1993) Lacunary statistical convergence, Pac. J. Math., **160**, 43–51.
[19] Altin Y., M. Et, R. Çolak (2006) Lacunary statistical and lacunary strongly convergence of generalized difference sequences of fuzzy numbers, Comp. Math. Appl., **52**(6–7), 1011–1020, https://doi.org/10.1016/j.camwa.2006.03.025.
[20] Kandemir H. Ş., E. Mikail, H. Çakalli (2023) On $S_{\alpha}^{\beta}(\theta, A, F)$-convergence and strong $N_{\alpha}^{\beta}(\theta, A, F)$-convergence, Boletim da Sociedade Paranaense de Matemática, **41**, 1–8, https://doi.org/10.5269/bspm.51415.

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