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ON COVERINGS BY MINKOWSKI BALLS IN THE PLANE AND A DUALITY

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Abstract

Lattice coverings in the real plane by Minkowski balls are studied. We exploit the duality of admissible lattices of Minkowski balls and inscribed convex symmetric hexagons of these balls. An explicit moduli space of the areas of these hexagons is constructed, giving the values of the determinants of corresponding covering lattices. Low and upper bounds for covering constants of Minkowski balls are given. The best known value of covering density of Minkowski ball is obtained. One conjecture and one problem are formulated.

Key words: lattice covering, Minkowski metric, Minkowski ball, hexagon, covering constant, covering density, thinnest covering

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1. Introduction. A system of equal balls in n-dimensional space is said to form a covering, if each point in space belongs to at least one ball of this system.

Let

$$D_p: \ |x|^p + |y|^p < 1, \ p \geq 1 \tag{1}$$

be (two-dimensional) Minkowski balls with boundary

$$C_p: \ |x|^p + |y|^p = 1, \ p \geq 1. \tag{2}$$

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Lattice coverings in the real plane by Minkowski balls with inscribed convex symmetric hexagons generated by admissible lattices of Minkowski balls are studied. The covering problem is dual to the packing problem. In different covering problems, this duality, apparently, can manifest itself in different ways. In this paper, duality is understood as the duality between admissible lattices having 3 pairs of points on the boundary D_p, and symmetric convex hexagons (possibly a quadrangle) corresponding to these lattices and determining the values of the determinants of the corresponding covering lattices.

Geometrically, the set of Minkowski balls is a solid tube of variable diameter with square sections at the ends $p = 1$, and $p = \infty$ which we call the limiting Minkowski balls. We use admissible lattices that have 3 pairs of points on the boundary of D_p. We construct an explicit moduli space (6) of areas of symmetric convex hexagons (possibly a quadrangle) which determine values of the determinants of corresponding covering lattices.

This construction and its consequences make it possible to obtain the best currently known lower bounds for covering constants of Minkowski balls. An estimate from above is also given. These give the corresponding estimation of the covering densities. The best known value of covering density of Minkowski ball is obtained (Proposition 7).

Our investigations connect with Minkowski conjecture [1–5, 12] and use results of its investigation. This conjecture was formulated by MINKOWSKI while studying a problem of Diophantine approximations for sums of powers of linear forms [1]. MORDELL [2] tested and confirmed this conjecture for $p = 4$. DAVIS [3] adjusted Minkowski conjecture for small values of $1 < p \leq p_0$ (see Theorems 2, 3). Minkowski (implicitly) and COHN [4] explicitly defined an appropriate moduli space for computing determinants of critical lattices. We follow these researchers, albeit in a slightly different sequence. Our research on the packing problem [14] was also used. Corresponding results and conjectures are stated in the simplest way in terms of geometric lattices and covering lattices [1, 7–9].

Based on these studies, one conjecture about the behaviour of the covering constants of Minkowski balls (see Conjecture 1) and one problem on belonging of the curve of maxima of covering constant to a family introduced in the work (see Problem 1) are formulated.

2. Two-dimensional Minkowski balls, admissible lattices and hexagons.

Proposition 1 ([1, 14]). *The volume of Minkowski ball D_p is equal to*

$$4\frac{\left(\Gamma\left(1+\frac{1}{p}\right)\right)^2}{\Gamma\left(1+\frac{2}{p}\right)}.$$

***2.1. Admissible lattices* [7, 8].** Let $\mathcal{R}$ be a set and Λ be a lattice with base

$$\{a_1, \dots, a_n\}$$

in $\mathbb{R}^n$.

A lattice Λ is *admissible* for body $\mathcal{R}$ ($\mathcal{R}$-*admissible*) if $\mathcal{R} \bigcap \Lambda = \varnothing$ or 0.

Let $\overline{\mathcal{R}}$ be the closure of $\mathcal{R}$. A lattice Λ is *strictly admissible* for $\overline{\mathcal{R}}$ ($\overline{\mathcal{R}}$-*strictly admissible*) if $\overline{\mathcal{R}} \bigcap \Lambda = \varnothing$ or 0.

Let

$$d(\Lambda) = |\det(a_1, \dots, a_n)|$$

be the determinant of Λ.

2.2. Hexagons and coverings. Every admissible lattice of D_p containing three pairs of points on the boundary of D_p defines a hexagon inscribed in D_p. We denote such a hexagon as $\mathcal{H}_p$, and call it a hexagon of the admissible lattice, briefly: *al-hexagon*.

Remark 1. In the limiting cases $p = 1$ and $p = \infty$ corresponding hexagons (following Fejes Toth, see [8]) $\mathcal{H}_p$ are quadrangles (see Proposition 3).

By specifying results from [8] to the case of Minkowski balls and al-hexagons $\mathcal{H}_p$ we have the next

Proposition 2. *Each Minkowski ball D_p contains an al-hexagon (possibly a quadrangle) $\mathcal{H}_p$ of maximum area. If this area is denoted by $\gamma_h(D_p)$ (or for symmetric convex inscribed al-hexagons $\mathcal{H}_p$ by $a(\mathcal{H}_p)$), then*

$$\Gamma(D_p) = \gamma_h(D_p). \tag{3}$$

For the density of thinnest lattice covering of D_p we get

$$\vartheta(D_p) = V(D_p)/\gamma_h(D_p). \tag{4}$$

Remark 2. Expression (3) is the **covering constant** of Minkowski ball D_p, and expression (4) is the **density** of the **thinnest lattice covering** within the framework of our considerations.

Example 1.

1. The lattice

   $$\Lambda_1^{(1)} = \{(1, 1), (2, 0)\}$$

   is the covering lattice by D_1. The limiting case of Minkowski balls with $p = 1$ gives an optimal covering (indeed a partition) of density 1.

2. The lattice

   $$\Lambda_\infty^{(1)} = \{(2, 2), (0, 2)\}$$

   is the covering lattice by D_∞. Limiting case of Minkowski balls for $p = \infty$ gives the optimal covering (indeed a partition) of density 1.

3. The lattice

$$\Lambda_2^{(0)} = \left\{ (\sqrt{3}, 0), \left(\frac{\sqrt{3}}{2}, \frac{3}{2} \right) \right\}$$

is the covering lattice by D_2. The lattice gives the optimal covering of density $\vartheta = 2\pi/3\sqrt{3} \approx 1.19$.

3. Moduli spaces, moduli problems and areas of inscribed hexagons. Moduli spaces are spaces of solutions of geometric classification problems and dynamics of varying objects in families [10, 11].

Example: moduli space for Legendre elliptic curve:

$$y^2 = x(x-1)(x-\lambda), \quad \lambda \in \mathbb{C}.$$

Moduli problem: Try to obtain solution for all objects of the moduli space or for objects of some subspace of the corresponding moduli space.

Currently, there is great interest in unifying problems of algebraic, arithmetic and geometric classification and dynamics through moduli spaces of object classes and studying related problems within such unifications. Various types of moduli spaces were presented and studied in the famous works of Eichler, Lazard, Shimura, Siegel, Igusa, Deligne–Mumford–Knudsen, Deligne–Rapoport, Katz–Mazur, Faltings, Drinfeld and others. But the author is not aware of applications of moduli spaces to covering problems.

In our present research the moduli space is real differentiable moduli space with exceptional points.

3.1. Solid Minkowski tube, Minkowski tube and metrics.

Definition 1. We will call (1) a solid Minkowski tube and (2) a Minkowski tube.

Remark 3. Expressions

$$(|x|^p + |y|^p)^{1/p} \tag{5}$$

are metrics. They also form for $|x| < a$, $|y| < b$, $a > 0$, $b > 0$ solid tubes of metric values for $1 \leq p \leq \infty$.

Proposition 3. *A Minkowski tube* (2) *is a real manifold which, for* $p = 1$ *and* $p = \infty$*, has exceptional points corresponding to the corners of the corresponding limit quadrangles.*

Proof. It follows from [1] and from the definition of Minkowski tube. □

The use of moduli spaces and the study of moduli problems require the adjustment of some well-known concepts.

Following Fejes Toth, a closed finite convex region bounded by at most six straight line segments will be called a hexagon.

This definition of a hexagon was mentioned by Bambah, Rogers and other famous covering researchers [6].

3.2. Inscribed hexagons of admissible lattices of Minkowski balls D_p and their moduli space.

Theorem 1. *The set of areas of the entire family of al-hexagons (possibly a quadrangle) $\mathcal{H}_p$ $(1 < p < \infty)$ inscribed in Minkowski balls D_p is parameterized by the function $A(\sigma, p)$ defining the corresponding moduli space* $\mathbf{A}$:

$$A(\sigma, p) = 3(\tau + \sigma)(1 + \tau^p)^{-\frac{1}{p}}(1 + \sigma^p)^{-\frac{1}{p}}. \tag{6}$$

In limiting cases we have the following areas of quadrangles: $p = 1$, $a(\mathcal{H}_1) = 2$; $p = \infty$, $a(\mathcal{H}_\infty) = 4$.

Proof. The area $a(\mathcal{H}_p)$ of an al-hexagon is given by

$$a(\mathcal{H}_p) = 2d\begin{pmatrix} x_1 & y_1 \\ x_1 - x_2 & y_1 + y_2 \end{pmatrix} + d\begin{pmatrix} -x_2 & y_2 \\ -x_1 & -y_1 \end{pmatrix} = 3(x_1 y_2 + x_2 y_1), \tag{7}$$

where $x_1 > x_2 \geq 0$, $y_1 \geq 0$, $y_2 > 0$ and $x_1^p + y_1^p = |-x_2|^p + y_2^p = (x_1 - x_2)^p + (y_1 + y_2)^p = 1$, $1 < p < \infty$.

The parameterization of admissible lattices (al) is implicitly given in [1] and explicitly specified in [4,5,12–14]. We specify the parameterization of al-hexagons. Let

$$0 \leq \tau < \sigma, \quad 0 \leq \tau \leq \tau_p.$$

τ_p is defined by the equation $2(1 - \tau_p)^p = 1 + \tau_p^p$, $0 \leq \tau_p < 1$.

$$1 \leq \sigma \leq \sigma_p, \quad \sigma_p = (2^p - 1)^{\frac{1}{p}}.$$

For inscribed in D_p a hexagon $\mathcal{H}_p$ with points

$$(x_1, y_1) = \left((1 + \tau^p)^{-\frac{1}{p}}, \tau(1 + \tau^p)^{-\frac{1}{p}}\right), \ (-x_2, y_2) = \left(-(1 + \sigma^p)^{-\frac{1}{p}}, \sigma(1 + \sigma^p)^{-\frac{1}{p}}\right),$$
$$(x_1 - x_2, y_1 + y_2) = \left((1 + \tau^p)^{-\frac{1}{p}} - (1 + \sigma^p)^{-\frac{1}{p}}, \tau(1 + \tau^p)^{-\frac{1}{p}} + \sigma(1 + \sigma^p)^{-\frac{1}{p}}\right).$$

This gives formula (6). The limiting cases follow from the proof of Minkowski conjecture ([12] or Theorem 2, see also 1. and 2. of Example 1). □

Definition 2. Moduli space (6) is the set of areas of the entire family of al-hexagons (possibly a quadrangle) $\mathcal{H}_p$ $(1 \leq p \leq \infty)$ inscribed in Minkowski balls D_p. Briefly we will call (6) the moduli space of hexagons.

Remark 4. The function domain of the function (6) is

$$\mathcal{M} : \infty > p > 1, \quad 1 \leq \sigma \leq \sigma_p = (2^p - 1)^{\frac{1}{p}}.$$

$\mathcal{M}$ has two limit points $p = 1$ and $p = \infty$.

Remark 5. The function (6) is upwardly convex (concave) in the real space with Cartesian coordinates (σ, p, A).

4. Sections of moduli space of hexagons. We will solve the problem of the thinnest covering of the real plane by Minkowski balls with hexagons $\mathcal{H}_p$ if we define a curve of maxima on an upwardly convex (concave) surface (6). Unfortunately, to date the author knows only a few points of maxima on this curve. In this regard, we present here a family of surfaces, give some intersections of (6) with these surfaces and formulate a (rough) hypothesis about the curve of maxima on (6).

Definition 3. Let

$$\sigma_{\alpha,p} = (2^p - 1)^{\frac{1}{\alpha p}}, \quad \alpha \geq 1 \tag{8}$$

be a family of curves parameterized by α with p varying from 1 to ∞ in real plane $\{\sigma, p\}$.

Remark 6. $\sigma_{\alpha,p} \neq \sigma_{p,\alpha}$.

Remark 7. Let us consider sections orthogonal to the plane $\{\sigma, p\}$ from curves $\sigma_{\alpha,p}$ to the moduli space $\mathbf{A}$. Such sections determine the covering $A\sigma_{\alpha,p}$ of the moduli space $\mathbf{A}$. Thus curves of the family (8) define ... family of curves

$$A(\alpha, p)\colon 3\left(\tau + (2^p - 1)^{\frac{1}{\alpha p}}\right)(1 + \tau^p)^{-\frac{1}{p}}\left(1 + \left((2^p - 1)^{\frac{1}{\alpha p}}\right)^p\right)^{-\frac{1}{p}} \tag{9}$$

on the function range (6). These curves are parameterized by α. For each such α, p varies from 1 to ∞ $(1 < p < \infty)$.

5. Estimates and bounds on covering constants. Here we estimate covering constants of Minkowski balls from above and from below. According to the results of Sas and Dovker (see [8]) in our case of Minkowski balls we have

Proposition 4. *Let $\mathcal{H}_p$ be the hexagon of maximum area inscribed in Minkowski ball D_p. Then $V(\mathcal{H}_p) = \gamma_h(D_p) \geq \frac{3\sqrt{3}}{2\pi} \cdot V(D_p)$.*

Remark 8. The trivial estimate from above of $V(\mathcal{H}_p)$ is given by the area $V(D_p)$ of Minkowski ball:

$$V(\mathcal{H}_p) \leq V(D_p).$$

Let estimate covering constants of Minkowski balls D_p from below on the base of results of the proof of Minkowski conjecture. To do this, we will use the results of the proof of the Minkowski conjecture (now theorem [12]) and the constructed moduli space (6). In this regard recall at first some definitions and results.

The infimum $\Delta(\mathcal{R})$ of determinants of all lattices admissible for $\mathcal{R}$ is called *the critical determinant* of $\mathcal{R}$; if there is no $\mathcal{R}$-admissible lattices, then puts $\Delta(\mathcal{R}) = \infty$.

A lattice Λ is *critical* if $d(\Lambda) = \Delta(\mathcal{R})$.

Theorem 2 ([12]).

$$\Delta(D_p) = \begin{cases} \Delta(p, 1), & 1 \leq p \leq 2, \ p \geq p_0, \\ \Delta(p, \sigma_p), & 2 \leq p \leq p_0. \end{cases}$$

Here p_0 is a real number that is defined unique by conditions $\Delta(p_0,\sigma_p)=\Delta(p_0,1)$, $2.57<p_0<2.58$, $p_0\approx 2.5725$.

Remark 9. We will call p_0 the Davis constant.

Corollary 1. *From Theorem 2 in notations* [12,13] *we have next expressions for critical determinants and their lattices*:

1. $\Delta_p^{(0)}=\Delta(p,\sigma_p)=\frac{1}{2}\sigma_p$;
2. $\sigma_p=(2^p-1)^{1/p}$;
3. $\Delta_p^{(1)}=\Delta(p,1)=4^{-\frac{1}{p}}\frac{1+\tau_p}{1-\tau_p}$;
4. $2(1-\tau_p)^p=1+\tau_p^p$, $0\le\tau_p<1$.

For their critical lattices respectively $\Lambda_p^{(0)}$, $\Lambda_p^{(1)}$ next conditions satisfy: $\Lambda_p^{(0)}$ and $\Lambda_p^{(1)}$ are two D_p-admissible lattices each of which contains three pairs of points on the boundary of D_p with the property that

- $(1,0)\in\Lambda_p^{(0)}$,
- $(-2^{-1/p},2^{-1/p})\in\Lambda_p^{(1)}$

(under these conditions the lattices are uniquely defined).

We specify the results of Theorem 2 for the moduli space (6).

Theorem 3.

$$\min(\mathbf{A}(\mathcal{H}_p))=\begin{cases}3\cdot 4^{-1/p}\frac{1+\tau_p}{1-\tau_p}, & 1\le p\le 2,\ p\ge p_0,\\ \frac{3}{2}\sigma_p, & 2\le p\le p_0.\end{cases} \tag{10}$$

To estimate the covering constants of Minkowski balls D_p from below, we define the inverse minimum $i-\min(\mathbf{A}(\mathcal{H}_p))$ for $\mathbf{A}(\mathcal{H}_p)$.

Definition 4.

$$i-\min(\mathbf{A})=i-\min(\mathbf{A}(\mathcal{H}_p))=\begin{cases}\frac{3}{2}\sigma_p, & 1\le p\le 2,\ p\ge p_0,\\ 3\cdot 4^{-1/p}\frac{1+\tau_p}{1-\tau_p}, & 2\le p\le p_0.\end{cases} \tag{11}$$

Example 2. In the case $p=3$ for moduli space $\mathbf{A}$ (see (6)) we have

$$\min(\mathbf{A}(\mathcal{H}_3))\approx 2.859,\ i-\min(\mathbf{A}(\mathcal{H}_3))\approx 2.870,\ \min(\mathbf{A}(\mathcal{H}_3))<i-\min(\mathbf{A}(\mathcal{H}_3)).$$

From Theorems 1, 3 with Definition 3 we have the next estimation from below of the covering constant $\Gamma(D_p)$.

Proposition 5.

$$\Gamma(D_p)\ge i-\min(\mathbf{A}). \tag{12}$$

Remark 10. For $p>2$ the estimate (12) is better than the estimate of Proposition 4.

6. Estimates and calculation of covering density. At first give the estimation of the thinnest lattice covering of D_p from above on the base of results by Sas and Dovker ([8]).

Proposition 6. *The density ϑ of any covering of $\mathbb{R}^2$ by Minkowski balls D_p satisfies*

$$\vartheta \leq \frac{2\pi}{3\sqrt{3}}. \tag{13}$$

Proof. By Proposition 2, $\Gamma(D_p) = \gamma_h(D_p)$; hence follows (13). □

Let us now examine the curves of the family (9) from the point of view of the coverage densities they determine. Here we do this only for one point on one curve. But even such example gives earlier unknown minimum density for Minkowski ball.

Take the curve from the family 9 defined by the function

$$A(2,p) = 3\left(\tau + (2^p - 1)^{\frac{1}{2p}}\right)(1+\tau^p)^{-\frac{1}{p}}\left(1 + \left((2^p-1)^{\frac{1}{2p}}\right)^p\right)^{-\frac{1}{p}}. \tag{14}$$

Proposition 7. *The convex symmetric hexagon defined by the point $p = 3$ on the curve* (14) *gives a covering density $\vartheta(D_3) \approx 1.0567$ by the Minkowski ball D_3, the minimum for known densities of Minkowski balls at $1 < p < \infty$.*

Proof.
$\sigma_{2,3} = (2^3 - 1)^{\frac{1}{2\cdot 3}} \approx 1.3830$;
$\tau_3 \approx 0.20406$, $0 \leq \tau \leq \tau_3$, for curve (14) take $\tau = 0.1200$;
$V(D_3) \approx 3.5200$, $\gamma_h(D_3) \approx 3.3310$.
So $\vartheta(D_3) = V(D_3)/\gamma_h(D_3) \approx 1.0567$. □

We state here one conjecture and one problem which arise naturally from our work.

Conjecture 1. *The curve of maxima of covering constants increases from $p = 1$ to $p = 2$ and decreases from $p = 2$ to $p = \infty$.*

Problem 1. Does the curve of maxima of covering constants belong to the family (9)?

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REFERENCES

[1] Minkowski H. (1907) Diophantische Approximationen, Leipzig, Teubner.
[2] Mordell L. J. (1941) Lattice points in the region $|Ax^4| + |By^4| \geq 1$, J. London Math. Soc., **16**, 152–156.

[3] Davis C. (1948) Note on a conjecture by Minkowski, J. London Math. Soc., **23**, 172–175.
[4] Cohn H. (1950) Minkowski's conjectures on critical lattices in the metric $\{|\xi|^p + |\eta|^p\}^{1/p}$, Annals of Math., **51**(2), 734–738.
[5] Watson G. (1953) Minkowski's conjecture on the critical lattices of the region $|x|^p + |y|^p \leq 1$, (I), (II), Jour. London Math. Soc., **28**(3, 4), 305–309, 402–410.
[6] Rogers C. (1964) Packing and Covering, Cambridge Univ. Press.
[7] Cassels J. W. S. (1997) An Introduction to the Geometry of Numbers, Springer, NY.
[8] Lekkerkerker C. G. (1969) Geometry of Numbers, North Holland, Amsterdam.
[9] Conway J. H., N. J. A. Sloane (1992) Sphere Packings, Lattices and Groups, Second Edition, Springer-Verlag, New York, Berlin.
[10] Katz N. M., B. Mazur (1985) Arithmetic Moduli of Elliptic Curves, Annals of Math. Studies, Volume 108, Princeton Univ. Press, Princeton.
[11] Harris J., I. Morrison (1998) Moduli of Curves, Springer-Verlag, New York.
[12] Glazunov N., A. Golovanov, A. Malyshev (1986) Proof of Minkowski's hypothesis about the critical determinant of $|x|^p+|y|^p < 1$ domain, Research in Number Theory 9, Notes of Scientific Seminars of LOMI, **151**, Leningrad, Nauka, 40–53.
[13] Glazunov N. (2016) On A. V. Malyshev's approach to Minkowski's conjecture concerning the critical determinant of the region $|x|^p + |y|^p < 1$ for $p > 1$, Chebyshevskii Sb., Volume 17, Issue 4, 185–193.
[14] Glazunov N. (2023) On packing of Minkowski balls, C. R. Acad. Bulg. Sci., **76**(3), 335–342.

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