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# ON THE DISTRIBUTION OF $\alpha p + \beta$ MODULO ONE FOR PRIMES OF TYPE $p = ar^2 + 1$

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**Abstract**

We study the distribution of fractional part of $\alpha p + \beta$ modulo 1, with irrational $\alpha$ and $p$ prime of the form $p = ar^2 + 1$.

**Key words:** distribution modulo one, primes in quadratic progressions
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**1. Introduction and statements of the result.** In 1923 Hardy and Littlewood [1] state the hypothesis:

**Conjecture 1.** *Suppose $a$, $b$, $c \in \mathbb{Z}$ with $a > 0$, $GCD(a, b, c) = 1$, $a + b$ and $c$ are not both even, and $D = b^2 - 4ac$ is not a square. Let $P_f(x)$ be the number of primes $p \le x$ of the form $p = f(n) = an^2 + bn + c$ with $n \in \mathbb{Z}$. Then*

$$P_f(x) \asymp GCD(2, a+b) \frac{\sigma(D) x}{\sqrt{a} \log x} \prod_{\substack{p|a,\, p|b \\ p>2}} \frac{p}{p-1},$$

*where*

$$\sigma(D) = \prod_{\substack{p \nmid a \\ p>2}} \left(1 - \frac{\left(\frac{D}{p}\right)}{p-1}\right).$$

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To date, there are numerous results related to this hypothesis. Assuming the extended Riemann hypothesis for $L$-functions on Hecke characters in 1952 Ankeny [2] proved that there are infinitely many primes of the form $x^2 + y^2$ with $y = O(\log x)$. In 1978 Iwaniec [3] established that there are infinitely many numbers of the form $n^2 + 1$ which have at most two prime divisors. Later Merikoski [4] showed that there are infinitely many numbers of the form $n^2 + 1$ with greatest prime factor at least $n^{1.279}$. Using Brun's sieve it can be proved that the density of primes up to $x$ which having the form $p = n^2 + 1$ is $O(\sqrt{x}/\log x)$. So almost all numbers of the form $n^2 + 1$ are composite.

In 2006, Baier and Zhao [5] proved that for $\varepsilon > 0$ there exist infinitely many primes of the form $p = am^2 + 1$ such that $a \le p^{5/9+\varepsilon}$. Later Matomäki [6] improved their result and proved that for $\varepsilon > 0$ there exist infinitely many primes of the form $p = aq^2 + 1$ such that $a \le p^{1/2+\varepsilon}$ and $q$ is prime.

In the present paper we consider another popular problem with primes. Let $\alpha$ be irrational real number, $\beta$ be real and let $\|x\| = \min\limits_{n\in\mathbb{Z}} |x - n|$. In 1947 Vinogradov [7] proved that if $0 < \theta < 1/5$ then there are infinitely many primes $p$ such that

$$\|\alpha p + \beta\| < p^{-\theta}.$$

Later the upper bound for $\theta$ was improved and the strongest published result is due Matomäki [8] with $\theta < 1/3$.

Our result is a hybrid between the two problems mentioned above. We shall prove the following

**Theorem 1.** *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\beta \in \mathbb{R}$ and let $0 < \theta < 1/108$. Then for any arbitrary small $\varepsilon > 0$ there are infinitely many primes $p$ such that $p = ar^2 + 1$ with $a$, $r \in \mathbb{N}$,*

$$a \le p^{2/3+4\theta+\varepsilon} \quad \text{and} \quad \|\alpha p + \beta\| < p^{-\theta}.$$

**2. Notation.** Let $x$ be a sufficiently large real number,

$$\delta = \delta(x) = x^{-\theta}, \quad K = \delta^{-1}\log^2 \quad y = x^{1/6-2\theta-\eta/2}, \quad \theta < \frac{1}{108}, \tag{1}$$

where $\eta$ is arbitrary small and positive number, which we will choose later. By $p$ we always denote prime. As usual $\varphi(n)$, $\Lambda(n)$, $\tau_k(n)$ are Euler's function, Mangoldt's function and the number of solutions of the equation $m_1 m_2 \cdots m_k = n$ in natural numbers $m_1, \ldots, m_k$, $\tau_2(n) = \tau(n)$ and

$$\psi(x, d, a) = \sum_{\substack{n\le x\\ n\equiv a(d)}} \Lambda(n).$$

With $\|y\|$ we denote the distance from $y$ to the nearest integer, $e(y) = e^{2\pi i y}$ and if $X < x \le 2X$, we will write $x \sim X$. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n(k)$. The letter $\varepsilon$ denotes an arbitrary small positive number, not the same in all appearances. For example this convention allows us to write $x^\varepsilon \log x \ll x^\varepsilon$.

**3. Some lemmas.**

**Lemma 1.** *Let $X \ge 1$ and $a$, $d \in \mathbb{N}$. Then*

$$\Bigg|\sum_{\substack{n\le X\\ n\equiv a\,(d)}} e(\alpha n)\Bigg| \ll \min\left(\frac{X}{d}, \frac{1}{\|\alpha d\|}\right).$$

**Proof.** See [9], ch.6, §2. □

**Lemma 2.** *Let $\varepsilon$ and $A$ be arbitrary positive constants. If $(\log x)^{A+1} \ll y \ll x^{2/9-\varepsilon}$, then*

$$\sum_{n\sim x}\Lambda(n+1)\sum_{\substack{q\sim y\\ q^2|n}} 1 = \frac{x}{2\zeta(2)y} + O\bigg(\frac{x}{y(\log x)^A}\bigg),$$

*where the $O$-constant depends only on $\varepsilon$ and $A$.*

**Proof.** See Lemma 10, [5]. □

**Lemma 3.** *Let $x$, $M$, $J \in \mathbb{R}^+$, $\mu$, $\zeta \in \mathbb{N}$ and $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ satisfy conditions*

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}, \quad a\in\mathbb{Z}, ;\ q\in\mathbb{N}, \quad (a,q)=1, \quad q\ge 1. \tag{2}$$

*Then for every arbitrary small $\varepsilon > 0$ the inequality*

$$\sum_{m\sim M}\tau_\mu(m)\sum_{j\sim J}\tau_\zeta(j)\min\left\{\frac{x}{m^2 j}, \frac{1}{\|\alpha m^2 j\|}\right\} \ll x^\varepsilon\bigg(MJ + \frac{x}{M^{3/2}} + \frac{x}{Mq^{1/2}} + \frac{x^{1/2}q^{1/2}}{M}\bigg)$$

*is fulfilled.*

**Proof.** See Lemma 8, [10]. □

**4. Auxiliary results.**

**Lemma 4.** *Let $x$, $M$, $J \in \mathbb{R}^+$, $\mu$, $\zeta \in \mathbb{N}$ and $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ satisfy conditions* (2)*. Then for any $\varepsilon > 0$ the inequality*

$$\sum_{m\sim M}\tau_\mu(m)\sum_{j\sim J}\tau_\zeta(j)\min\left\{\frac{x}{m^4 j}, \frac{1}{\|\alpha m^4 j\|}\right\} \le x^\varepsilon\bigg(MJ + \frac{x}{M^{\frac{25}{8}}} + \frac{x}{M^3 q^{\frac{1}{8}}} + \frac{x^{\frac{7}{8}}q^{\frac{1}{8}}}{M^3}\bigg)$$

*is fulfilled.*

**Proof.** Our proof is similar to the proof of Lemma 8, [10]. Let

$$H = \frac{x}{M^4 J}. \tag{3}$$

If $H \le 2$, then using that $\tau_k(n) \ll_{k,\varepsilon} n^\varepsilon$ we get

$$G \ll x^\varepsilon MJ. \tag{4}$$

So we can assume that $H > 2$. It is obvious that

$$G \ll x^\varepsilon \sum_{m\sim M} \sum_{j\sim J} \min\left\{ \frac{x}{m^4 j}, \frac{1}{\|\alpha m^4 j\|} \right\}.$$

We apply the Fourier expansion to function $\min\left\{ \frac{x}{m^4 j}, \frac{1}{\|\alpha m^4 j\|} \right\}$ and get

$$\min\left\{ \frac{x}{m^4 j}, \frac{1}{\|\alpha m^4 j\|} \right\} = \sum_{0<|h|\le H^2} w(h) e(\alpha m^4 jh) + O(\log x),$$

where

$$w(h) \ll \min\left\{ \log H, \frac{H}{|h|} \right\}. \tag{5}$$

Then

$$|G| \ll x^\varepsilon \sum_{0<|h|\le H^2} |w(h)| \sum_{j\sim J} \left| \sum_{m\sim M} e(\alpha m^4 jh) \right| + MJ \log x. \tag{6}$$

So if

$$G(H_0) = \sum_{h\sim H_0} \sum_{j\sim J} \left| \sum_{m\sim M} e(\alpha m^4 jh) \right|.,$$

then using (5) we have

$$G \ll x^\varepsilon \left( MJ + \max_{1\le H_0\le H_1} G(H_0) + \max_{H_1<H_0\le H^2} \frac{H}{H_0} G(H_0) \right). \tag{7}$$

We shall evaluate the sum $G(H_0)$. Applying the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} G^2(H_0) &\ll x^\varepsilon H_0 J \sum_{h\sim H_0} \sum_{j\sim J} \left| \sum_{m\sim M} e(\alpha m^4 jh) \right|^2 \\ &\ll x^\varepsilon H_0 J \sum_{h\sim H_0} \sum_{j\sim J} \sum_{m_1\sim M} \sum_{m_2\sim M} e\big(\alpha(m_2^4 - m_1^4) jh\big). \end{aligned}$$

Substituting $m_2 = m_1 + t$, where $0 \le |t| \le M$ we get

$$G^2(H_0) \ll x^\varepsilon \Big( H_0^2 J^2 M + H_0 J G_1(H_0) \Big), \tag{8}$$

where

$$G_1(H_0) = \sum_{h\sim H_0}\sum_{j\sim J}\sum_{0<|t|<M}\Bigg|\sum_{m_1\sim M} e\big(\alpha(4m_1^3t+6m_1^2t^2+4m_1t^3)jh\big)\Bigg|.$$

Applying again the Cauchy–Schwarz inequality we obtain

$$G_1^2(H_0) \ll H_0JM\sum_{h\sim H_0}\sum_{j\sim J}\sum_{0<|t|<M}\sum_{m_2\sim M} \times \sum_{m_1\sim M} e\big(\alpha(4(m_2^3-m_1^3)t+6(m_2^2-m_1^2)t^2+4(m_2-m_1)t^3)jh\big).$$

Substituting $m_2 = m_1+\ell$, where $0\le|\ell|\le M$ we get

$$G_1^2(H_0) \ll H_0^2J^2M^3 + H_0JMG_2(H_0), \tag{9}$$

where

$$G_2(H_0) = \sum_{h\sim H_0}\sum_{j\sim J}\sum_{0<|t|<M}\sum_{0<|\ell|<M}\Bigg|\sum_{m_1\sim M} e\big(12\alpha(m_1^2t\ell+m_1\ell^2t+m_1\ell t^2)jh\big)\Bigg|.$$

Applying again the Cauchy–Schwarz inequality we obtain

$$G_2^2(H_0) \ll H_0JM^2\sum_{h\sim H_0}\sum_{j\sim J}\sum_{0<|t|<M}\sum_{0<|\ell|<M}\sum_{m_2\sim M} \times \sum_{m_1\sim M} e\big(12\alpha((m_2^2-m_1^2)t\ell+6(m_2-m_1)\ell^2t+(m_2-m_1)\ell t^2)jh\big).$$

Substituting $m_2 = m_1+z$, where $0\le|z|\le M$ we get

$$G_2^2(H_0) \ll H_0^2J^2M^5 + H_0JM^2\sum_{h\sim H_0}\sum_{j\sim J}\sum_{0<|t|<M}\sum_{0<|\ell|<M}\sum_{0<|z|<M}\Bigg|\sum_{m_1\sim M} e\big(24\alpha m_1zt\ell jh\big)\Bigg|.$$

Let $u = 24tz\ell zjh$. Then using Lemma 1 we obtain

$$G_2^2(H_0) \ll H_0^2J^2M^5 + H_0JM^2\sum_{u\le 24H_0JM^3}\tau_6(u)\min\left\{\frac{H_0JM^4}{u},\ \frac{1}{\|\alpha u\|}\right\}$$

$$\ll x^\varepsilon\bigg(H_0^2J^2M^5+\frac{H_0^2J^2M^6}{q}+H_0JM^2q\bigg). \tag{10}$$

From (8) we obtain

$$G(H_0) \ll x^\varepsilon\bigg(H_0JM^{7/8}+\frac{H_0JM}{q^{\frac{1}{8}}}+H_0^{\frac{7}{8}}J^{\frac{7}{8}}M^{\frac{1}{2}}q^{\frac{1}{8}}\bigg).$$

Choosing $H_0 = H$ from (7), (8), (9), (10), (3) and (4) we get

$$G \ll x^{\varepsilon}\left(JM + \frac{x}{M^{25/8}} + \frac{x}{M^3 q^{1/8}} + \frac{x^{\frac{7}{8}} q^{\frac{1}{8}}}{M^3}\right). \tag{11}$$

□

**5. Proof of Theorem 1.** Let $s(n) = n/r^2$, where $n \in \mathbb{N}$ and $r^2$ is the largest square dividing $n$. It is clear that $s(n) = 1$ if and only if $n$ is a perfect square. Let $p$ be a prime. Then $s(p-1) \le p^{2/3+4\theta+\eta}$ if and only if there exists $r \in \mathbb{N}$ such that $r^2 | p-1$ and $r \ge (p-1)^{1/6-2\theta-\eta/2}$. Therefore the number of primes $p \sim x-1$ such that $s(p-1) \le p^{2/3+4\theta+\eta}$ and $\|\alpha p + \beta\| < p^{-\theta}$ is greater than $\Gamma(x)/\log x$ with

$$\Gamma(x) = \sum_{\substack{n\sim x-1 \\ \|\alpha n+\beta\|<n^{-\theta}}} \frac{\Lambda(n)}{\tau(n-1)} \sum_{\substack{r\ge y \\ r^2|n-1}} 1 - C_2 x^{1/2+\varepsilon},$$

where $y = x^{1/6-2\theta-\eta/2}$ and the residual member has been received from addends of the form $n = s^k$, $k \ge 2$ and $s$-prime. Using that $\tau(n) \ll n^{\varepsilon/2}$ we get

$$\begin{aligned}\Gamma(x) &\ge C_1 x^{-\varepsilon/2} \sum_{\substack{n\sim x-1 \\ \|\alpha n+\beta\|<n^{-\theta}}} \Lambda(n) \sum_{\substack{r\ge y \\ r^2|n-1}} 1 - C_2 x^{1/2+\varepsilon} \\ &\ge C_1 x^{-\varepsilon/2} \sum_{\substack{n\sim x-1 \\ \|\alpha n+\beta\|<n^{-\theta}}} \Lambda(n) \sum_{\substack{r\sim y \\ r^2|n-1}} 1 - C_2 x^{1/2+\varepsilon}.\end{aligned} \tag{12}$$

As in [11] we take a periodic with period 1 function such that

$$\begin{aligned} 0 < \chi(t) < 1 \quad &\text{if} \quad -\delta < t < \delta; \\ \chi(t) = 0 \quad &\text{if} \quad \delta \le t \le 1-\delta, \end{aligned}$$

which has a Fourier series

$$\chi(t) = \delta + \sum_{|k|>0} c(k)e(kt), \tag{13}$$

with coefficients satisfying

$$c(0) = \delta, \quad c(k) \ll \delta \text{ for all } k, \quad \sum_{|k|>K} |c(k)| \ll x^{-1} \tag{14}$$

and $\delta$ and $K$ satisfying the conditions (1). The existence of such a function is a consequence of a well-known lemma of Vinogradov (see [9], ch. 1, §2). Then from (12) we get

$$\Gamma(x) \ge C_1 x^{-\varepsilon/2} \Gamma_1(x) - C_2 x^{1/2+\varepsilon}, \tag{15}$$

where

$$\Gamma_1(x)=\sum_{n\sim x-1}\Lambda(n)\chi(\alpha n+\beta)\sum_{\substack{r\sim y\\ r^2|n-1}}1.$$

From the Fourier expansion (13) of $\chi(t)$ we get

$$\Gamma_1(x)=\delta\bigg(\Gamma_2(x)+\Gamma_3(x)+O\Big(\frac{1}{y}\Big)\bigg), \tag{16}$$

where

$$\Gamma_2(x)=\sum_{r\sim y}\sum_{\substack{n\sim x\\ n\equiv 1(r^2)}}\Lambda(n)\quad\text{and}\quad\Gamma_3(x)=\sum_{0<|k|\leq K}c(k)\sum_{r\sim y}\sum_{\substack{n\sim x\\ n\equiv 1(r^2)}}e(\alpha kn)\Lambda(n).$$

Here we have laid $c(k):=c(k)e(\beta k)$.

From Lemma 2 we have

$$\Gamma_2(x)=\frac{x}{2\zeta(2)y}+O\bigg(\frac{x}{y(\log x)^A}\bigg). \tag{17}$$

***5.1. Estimate of the amount $\mathbf{\Gamma_3(x)}$.*** First we decompose the sum $\Gamma_3(x)$ into $O(\log x)$ sums of type

$$W=\sum_{k\sim K_0}c(k)\sum_{r\sim y}\sum_{\substack{n\sim x\\ n\equiv 1(r^2)}}\Lambda(n)e(\alpha kn),$$

where

$$1\leq K_0\leq K/2. \tag{18}$$

Then by Vaughan's identity we can decompose the sum $W$ into $O(\log x)$ type I sums

$$W_1=\sum_{k\sim K_0}c(k)\sum_{r\sim y}\sum_{m\sim M}a(m)\sum_{\substack{\ell\sim L\\ m\ell\equiv 1\,(r^2)}}e(\alpha m\ell k),$$

$$W_1'=\sum_{k\sim K_0}c(k)\sum_{r\sim y}\sum_{m\sim M}a(m)\sum_{\substack{\ell\sim L\\ m\ell\equiv 1\,(r^2)}}e(\alpha m\ell k)\log\ell,$$

where $ML\sim x$, $M\leq x^{1/3}$ and into $O(\log x)$ type II sums

$$W_2=\sum_{k\sim K_0}c(k)\sum_{r\sim y}\sum_{m\sim M}a(m)\sum_{\substack{\ell\sim L\\ m\ell\equiv 1\,(r^2)}}b(\ell)e(\alpha m\ell k),$$

where $ML\sim x$, $x^{1/3}\leq M\leq x^{2/3}$ and $a(m)\ll\tau(m)\log m$, $b(l)\ll\tau(l)\log l$.

First we estimate type I sums rather straightforwardly. We have

$$W_1 \ll x^{\varepsilon} \sum_{k\sim K_0} \sum_{r\sim y} \sum_{m\sim M} \Bigg| \sum_{\substack{\ell\sim L\\ m\ell\equiv 1\,(r^2)}} e(\alpha m\ell k)\Bigg|.$$

As $L > x^{2/3} > y^2 \geq r^2$ we get $\ell = f + r^2 t$, where $f = f(m, r)$. Using Lemma 1 we obtain

$$\begin{aligned} W_1 &\ll x^{\varepsilon} \sum_{k\sim K_0} \sum_{r\sim y} \sum_{m\sim M} \Bigg| \sum_{t\sim \frac{L}{r^2}} e(\alpha m t r^2 k)\Bigg| \\ &\ll x^{\varepsilon} \sum_{k\sim K_0} \sum_{r\sim y} \sum_{m\sim M} \min\left\{\frac{L}{r^2}\, \frac{1}{\|\alpha m r^2 k\|}\right\} \\ &\ll x^{\varepsilon} \sum_{u\sim K_0 M} \tau(u) \sum_{r\sim y} \min\left\{\frac{xK_0}{ur^2}\, \frac{1}{\|\alpha u r^2\|}\right\}. \end{aligned}$$

Applying Lemma 3 and bearing in mind (18) we get

$$W_1 \ll x^{\varepsilon}\left(y x^{1/3} K + \frac{xK}{y^{3/2}} + \frac{xK}{y q^{1/2}} + \frac{x^{1/2}K^{1/2}q^{1/2}}{y}\right). \tag{19}$$

Now we will estimate the sum $W_2$. It is enough to consider the case

$$x^{1/3} \leq L \leq x^{1/2} \quad , \quad x^{1/2} \leq M \leq x^{2/3}.$$

We start by applying the Cauchy–Schwarz inequality, obtaining

$$W_2^2 \leq x^{1+\varepsilon}\Big(xMK^2 + W_{21}\Big), \tag{20}$$

where

$$W_{21} = yMK_0 \sum_{k\sim K_0} \sum_{r\sim y} \sum_{m\sim M} \sum_{\substack{\ell_1,\ell_2\sim L\\ m\ell_1\equiv 1\,(r^2)\\ m\ell_2\equiv 1\,(r^2)\\ \ell_1\neq \ell_2}} b(\ell_1)b(\ell_2)e(\alpha m(\ell_1-\ell_2)k).$$

If $\ell_1 \neq \ell_2$ from $m\ell_i \equiv 1\,(r^2)$ follows $\ell_1 \equiv \ell_2\,(r^2)$. So $\ell_1 = \ell_2 + tr^2$ with $t \leq \dfrac{L}{y^2}$ and using Lemma 1 we receive

$$W_2^2 \ll x^{\varepsilon} yMK_0 \sum_{k\sim K_0} \sum_{r\sim y} \sum_{\ell_2\sim L} \sum_{t\leq \frac{L}{y^2}} \Bigg| \sum_{\substack{m\sim M\\ m\ell_2\equiv 1\,(r^2)}} e(\alpha m t r^2 k)\Bigg|$$

$$\ll x^{\varepsilon} xyK_0 \sum_{k\sim K_0} \sum_{r\sim y} \sum_{t\le \frac{L}{y^2}} \min\left\{\frac{M}{r^2}, \frac{1}{\|\alpha r^4 tk\|}\right\}.$$

By substituting $u = tk \ll \dfrac{LK_0}{y^2}$ it follows that

$$W_2^2 \ll x^{\varepsilon} xyK_0 \sum_{u\ll \frac{LK_0}{y^2}} \tau(u) \sum_{r\sim y} \min\left\{\frac{xK_0}{r^4 u}, \frac{1}{\|\alpha r^4 u\|}\right\}.$$

Using Lemma 4, (20) and (18) we get

$$W_2 \ll x^{\varepsilon}\left(x^{\frac{3}{4}}K + \frac{xK}{y^{\frac{17}{16}}} + \frac{xK}{yq^{\frac{1}{8}}} + \frac{x^{\frac{15}{16}}K^{\frac{15}{16}}q^{\frac{1}{16}}}{y}\right). \tag{21}$$

From (19), (21) and (5) follows

$$\Gamma_3 \ll x^{\varepsilon}\left(x^{\frac{3}{4}}K + \frac{xK}{y^{\frac{17}{16}}} + \frac{xK}{yq^{\frac{1}{8}}} + \frac{x^{\frac{15}{16}}K^{\frac{15}{16}}q^{\frac{1}{16}}}{y}\right). \tag{22}$$

From the denominators $q_1 < q_2 < \cdots$ of approximation of irrational $\alpha$ we choose $x_i$ such that the equality $\dfrac{xK}{y} = q$ is fulfilled. We take $\theta = \dfrac{1}{108} - 2\varepsilon$ and $\eta = 8\varepsilon$. So $y = x^{1/6-2\theta-4\varepsilon}$ and we get a sequence

$$x_1 < x_2 < x_3 < \cdots \to \infty$$

such that

$$\Gamma_3 \ll \frac{x^{1-\varepsilon}}{y}. \tag{23}$$

Hence from (15), (16), (17) and (23) we get

$$\Gamma \ge C_1 x_i^{-\varepsilon/2}\delta\left(\frac{x_i}{2\zeta(2)y_i} + O\left(\frac{x_i^{1-\varepsilon}}{y_i}\right)\right)$$

and the proof is complete.

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