

NOVEL FIXED POINT THEOREMS IN PARTIALLY
ORDERED MODULAR METRIC SPACES
WITH SOME APPLICATIONS

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Abstract

This paper introduces new types of the generalized C -condition for mappings in partially ordered modular metric spaces. For such conditions, the fixed point theorems are stated and proved. Some examples with graphical representation are given to support our findings. An integral equation is considered in the given modular metric space. The existence and also uniqueness of a solution for this equation are proved. These outcomes extend the fixed point results for the generalized C -condition to partially ordered modular metric spaces.

Key words: fixed point theorem, modular metric space, C -condition

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1. Introduction. Once Banach came up with the Banach contraction principle in 1922, the fixed point theory became a fascinating topic. Its applications have also been used in many areas since then. Furthermore, distinct types of this principle have been studied and extended in different spaces.

NAKANO [1] introduced the concept of modular spaces (MS) in 1950. This has led mathematicians to work on expanding the fixed point results attained on metric spaces to MS . In addition, MUSIELAK and ORLICZ [2] explored MS in detail and gave some theories about them. Later functionally modular spaces were introduced by KOZŁOWSKI [3]. For other results about fixed point findings on MS , see the paper of KUAKET and KUMAM [4] and the references therein.

In 2010, CHISTYAKOV [5] brought forward the concept of modular metric spaces (*MMS*). He gave the definition and fundamentals of modular metric (*MM*) [5]. In [6], the fixed point results for the contractions on *MMS* were given for the first time. Following this work, the study of fixed point theory in *MMS* began. MONGKOLKEHA and KUMAM [7], AKSOY et al. [8], EGE and ALACA [9], ABDON [10] and KARAPINAR et al. [11] put forth valuable fixed point results for the contractive mappings on *MMS*. These papers point out the importance and utilization of *MMS* in the literature.

We begin with the definition of the *C*-condition on usual metric spaces. SUZUKI [12] was the first one to define the *C*-condition as in the following.

Definition 1. Let G be a self mapping on a metric space (Ω, d) . G is said to yield the *C*-condition if for each $\tau_0, \tau_1 \in \Omega$

$$(1) \quad \frac{1}{2}d(\tau_0, G\tau_0) \leq d(\tau_0, \tau_1) \implies d(G\tau_0, G\tau_1) \leq d(\tau_0, \tau_1).$$

Then, he gave the following theorem.

Theorem 2. Assume $G: \Omega \rightarrow \Omega$ is a mapping where the metric space (Ω, d) is compact. If for each $\tau_0, \tau_1 \in \Omega$

$$(2) \quad \frac{1}{2}d(\tau_0, G\tau_0) \leq d(\tau_0, \tau_1) \implies d(G\tau_0, G\tau_1) \leq d(\tau_0, \tau_1),$$

is provided, then G has a fixed point.

Extending the above idea, POPESCU [13] proved the following theorem.

Theorem 3. Suppose that the metric space (Ω, d) is complete and $G: \Omega \rightarrow \Omega$ is a self-mapping. Suppose that G satisfies

$$(3) \quad \frac{1}{2}d(\tau_0, G\tau_0) \leq d(\tau_0, \tau_1) \implies d(G\tau_0, G\tau_1) \leq Ad(\tau_0, \tau_1) \\ + B[d(\tau_0, G\tau_0) + d(\tau_1, G\tau_1)] + C[d(\tau_0, G\tau_1) + d(\tau_1, G\tau_0)],$$

where $A + 2B + 2C = 1$ and $A \geq 0, B, C > 0$. Then, the fixed point of G is unique.

Definition 4. Consider the continuous function ψ , defined on $[0, \infty)$. If the following hold:

- Ψ_1 . ψ is nondecreasing,
- Ψ_2 . $\psi(t) = 0 \iff t = 0$,

then it is said that ψ is an altering distance function. Ψ denotes the set of such functions.

Definition 5 ([14]). Consider a mapping $G: \Omega \rightarrow \Omega$ on the given metric space (Ω, d) . If for each $\tau_0, \tau_1 \in \Omega$ the following holds

$$(4) \quad \frac{1}{2}d(\tau_0, G\tau_0) \leq d(\tau_0, \tau_1) \implies \psi(d(G\tau_0, G\tau_1)) \leq \beta(d(\tau_0, \tau_1)),$$

where β is defined and continuous on $[0, \infty)$ and ψ is in Ψ , then it is said that G satisfies C_β^ψ -condition.

GUPTA et al. [14] have given the above C -condition for (ψ, β) -mappings and obtained fixed point results for them.

As we work in partially ordered modular metric spaces ($POMMS$), it is beneficial to give a brief summary about investigating the fixed point in partially ordered spaces (POS). GUO and LAKSHMIKANTHAM [15] took the lead to examine the fixed points in POS . Moreover, BHASKAR and LAKSHMIKANTHAM [16] gave important results in such spaces. Another fundamental work was done by RAN and REURINGS [17]. As an application, an integral equation on a functional modular space in which the map is of contraction type is proposed for the first time by KHAMSI et al. [18].

Motivated by all the above, we focus on the fixed point results about generalized C -condition in $POMMS$. This paper introduces two types of generalized C -condition in MMS , and also some fixed point theorems and results on the given modular space for mappings yielding these conditions. Moreover, we give some examples and demonstrate both the existence and also uniqueness of the solutions. As an application, the existence and also uniqueness for a solution of the considered integral equation is examined.

2. Fundamentals of modular spaces. Let Ω be nonempty set and take a function m as $m: (0, \infty) \times \Omega \times \Omega \rightarrow [0, \infty]$. For $\eta > 0$, m can be written as $m(\eta, \tau_1, \tau_2) = m_\eta(\tau_1, \tau_2)$ for all $\eta > 0$ and $\tau_1, \tau_2 \in \Omega$ where $m_\eta: \Omega \times \Omega \rightarrow [0, \infty]$. For more detail on the topic, see [19].

Definition 6 (See [5]). Presume that the following hold

$$(5) \quad \begin{aligned} m1) \quad & \tau_0 = \tau_1 \iff m_\eta(\tau_0, \tau_1) = 0, \\ m2) \quad & m_\eta(\tau_0, \tau_1) = m_\eta(\tau_1, \tau_0), \\ m3) \quad & m_{\eta+\mu}(\tau_0, \tau_1) \leq m_\eta(\tau_0, \tau_2) + m_\mu(\tau_2, \tau_1) \end{aligned}$$

for all $\tau_0, \tau_1, \tau_2 \in \Omega$, and $\eta, \mu > 0$. Then, m_η is called a metric modular (MM) on Ω .

Definition 7 (See [5]). Function m is said to be strict on Ω provided that for $\tau_1, \tau_2 \in \Omega$ with $\tau_1 \neq \tau_2$, $m_\eta(\tau_1, \tau_2) > 0$ for all $\eta > 0$, or equivalently, if $m_\eta(\tau_1, \tau_2) = 0$ for some $\eta > 0$, then $\tau_1 = \tau_2$.

In particular, $m_\eta(\tau_0, \tau_1) = d(\tau_0, \tau_1)/\eta$ and $m_\eta(\tau_0, \tau_1) = d(\tau_0, \tau_1)$ are strict modular metrics where $d(\tau_0, \tau_1)$ is a usual metric.

Definition 8 (See [5]). Take an MM m on Ω and $\tau_0 \in \Omega$. The following are MMS around τ_0 :

$$(6) \quad \begin{aligned} \Omega_m &= \Omega_m(\tau_0) = \{\tau \in \Omega: m_\eta(\tau, \tau_0) \rightarrow 0 \text{ as } \eta \rightarrow \infty\}, \\ \Omega_m^* &= \Omega_m^*(\tau_0) = \{\tau \in \Omega: \exists \eta = \eta(\tau) > 0 \text{ such that } m_\eta(\tau, \tau_0) < \infty\}. \end{aligned}$$

Lemma 9 (See [6]). For any pseudomodular m on Ω , the MMS given in (6) are closed w.r.t. m -convergent. Moreover, the limit of any strict modular is unique.

Note that if $\lim_{n \rightarrow \infty} m_\eta(h_n, h) = 0$ for some $\eta > 0$, then $\lim_{n \rightarrow \infty} m_\nu(h_n, h) = 0$ for all $\nu < \eta$.

3. Main results. In this section, we consider the generalized C -condition in MMS proposed by GÜNDOĞDU et al. [20] and also prove some theorems about fixed points for the mentioned conditions in partially ordered complete MMS .

Definition 10 (See [20]). Let m be an MM on $(\Omega_m, \preceq)$. Any self-mapping $G: \Omega_m \rightarrow \Omega_m$ is named as a generalized C -condition of type I provided that G holds for the following

$$(7) \quad \frac{1}{2}m_\eta(\tau_0, G\tau_0) \leq m_\eta(\tau_0, \tau_1) \implies \\ \psi(m_\eta(G\tau_0, G\tau_1)) \leq \beta \left(\max \left\{ m_\eta(\tau_0, \tau_1), \frac{m_\eta(\tau_0, G\tau_0)(1 + m_\eta(\tau_1, G\tau_1))}{1 + m_\eta(\tau_0, \tau_1)}, \right. \right. \\ \left. \left. \frac{m_\eta(\tau_1, G\tau_0)(1 + m_\eta(\tau_1, G\tau_0))}{1 + m_\eta(\tau_0, \tau_1)} \right\} \right)$$

for all $\tau_0, \tau_1 \in \Omega$ with $\tau_0 \preceq \tau_1$. Here, β is defined and continuous on $[0, \infty)$, and ψ is in Ψ .

Definition 11 (See [20]). Let m be an MM on $(\Omega_m, \preceq)$. Any self-mapping $G: \Omega_m \rightarrow \Omega_m$ is named as a generalized C -condition of type II provided that G holds for the following

$$(8) \quad \frac{1}{2}m_\eta(\tau_0, G\tau_0) \leq m_\eta(\tau_0, \tau_1) \implies \\ \psi(m_\eta(G\tau_0, G\tau_1)) \leq \beta \left(\max \left\{ m_\eta(\tau_0, \tau_1), \frac{m_\eta(\tau_0, G\tau_0)(1 + m_\eta(\tau_1, G\tau_1))}{1 + m_\eta(\tau_0, \tau_1)}, \right. \right. \\ \left. \left. \frac{m_\eta(\tau_1, G\tau_1)m_\eta(\tau_0, G\tau_0)}{1 + m_\eta(G\tau_0, G\tau_1)}, \frac{m_\eta(\tau_1, G\tau_1)m_\eta(\tau_1, G\tau_0)}{1 + m_\eta(\tau_1, G\tau_0) + m_\eta(\tau_0, G\tau_1)} \right\} \right)$$

for all $\tau_0, \tau_1 \in \Omega$ with $\tau_0 \preceq \tau_1$. Here, β is defined and continuous on $[0, \infty)$, and ψ is in Ψ .

Note that a partially ordered m -complete MMS $(\Omega_m, m, \preceq)$ is an m -complete MMS (Ω_m, m) endowed with a partial order ' $\preceq$ ' on Ω_m .

Now, we give our main theorems for mappings satisfying the C -condition of type I and II on $POMMS$.

Theorem 12. Let an MM m be strict on Ω and the triple $(\Omega_m, m, \preceq)$ be an m -complete partially ordered modular metric space w.r.t. ' $\preceq$ '. Suppose that $G: \Omega_m \rightarrow \Omega_m$ is satisfying the generalized C -condition of type I with $\psi(\tau) > \beta(\tau)$ for all $\tau > 0$. Assume that there exists $w \in \Omega$, comparable to u and v for any $u, v \in \Omega$. In addition, if a point $u_0 \in \Omega$ holds $u_0 \preceq Gu_0$, then G has a unique fixed point that belongs to Ω_m .

Proof. Take $u_0 \in \Omega_m$ satisfying $u_0 \preceq Gu_0$. Then, consider a sequence $\{u_n\} \in \Omega_m$ as $u_n = G^n u_0$ for all $n \in N$. If, for any $n \in N$, $u_n = u_{n+1}$ holds, then u_n would be the fixed point. Hence, for all $n \in N$ u_n cannot be equal to u_{n+1} .

Let us put $\tau_0 = u_n$ and $\tau_1 = u_{n+1}$ in (7). Then we have

$$(9) \quad \frac{1}{2}m_\eta(u_n, Gu_n) = \frac{1}{2}m_\eta(u_n, u_{n+1}) \leq m_\eta(u_n, u_{n+1}) \implies$$

$$\psi(m_\eta(Gu_n, Gu_{n+1})) = \psi(m_\eta(u_{n+1}, u_{n+2})) \leq \beta \left(\max \left\{ m_\eta(u_n, u_{n+1}), \right. \right.$$

$$\left. \left. \frac{m_\eta(u_n, u_{n+1})(1 + m_\eta(u_{n+1}, u_{n+2}))}{1 + m_\eta(u_n, u_{n+1})}, \frac{m_\eta(u_{n+1}, u_{n+1})(1 + m_\eta(u_{n+1}, u_{n+2}))}{1 + m_\eta(u_n, u_{n+1})} \right\} \right).$$

Now, we assume that $m_\eta(u_{n+1}, u_{n+2}) > m_\eta(u_n, u_{n+1})$. In this case, from (9) we get

$$(10) \quad \psi(m_\eta(u_{n+1}, u_{n+2})) \leq \beta \left(\max \left\{ m_\eta(u_n, u_{n+1}), \frac{m_\eta(u_n, u_{n+1})(m_\eta(u_{n+1}, u_{n+2}))}{m_\eta(u_n, u_{n+1})} \right\} \right)$$

$$\leq \beta(\max\{m_\eta(u_n, u_{n+1}), (m_\eta(u_{n+1}, u_{n+2}))\}).$$

Since $m_\eta(u_{n+1}, u_{n+2}) > m_\eta(u_n, u_{n+1})$, we have

$$\psi(m_\eta(u_{n+1}, u_{n+2})) \leq \beta(m_\eta(u_{n+1}, u_{n+2})),$$

which is a contradiction. Thus, $m_\eta(u_{n+1}, u_{n+2}) \leq m_\eta(u_n, u_{n+1})$. Since the sequence $\{m_\eta(u_n, u_{n+1})\} \geq 0$ is non-increasing, it has a limit, say

$$\lim_{n \rightarrow \infty} m_\eta(u_n, u_{n+1}) = L \geq 0.$$

Taking limit of both sides in (10) yields $\psi(L) \leq \beta \left(\max \left\{ L, \frac{L}{1+L}(1+L) \right\} \right) = \beta(L)$, which gives a contradiction. Hence, $L = 0$, that is, $\lim_{n \rightarrow \infty} m_\eta(u_n, u_{n+1}) = 0$.

Now, we show that the sequence $\{u_n\}$ is m -Cauchy. Assume that $\{u_n\}$ is not an m -Cauchy sequence. By the assumption, for $\epsilon > 0$, two subsequences $\{h_{u_i}\}$ and $\{u_{m_i}\}$ of $\{h_n\}$ satisfying $n_l > m_l > l$ can be found such that n_l is the smallest index for which

$$(11) \quad m_\eta(u_{m_l}, u_{n_l}) \geq \epsilon, \quad m_\eta(u_{m_l}, u_{n_{l-1}}) < \epsilon$$

for all $\eta > 0$. Then, the following holds

$$(12) \quad \epsilon \leq m_{4\eta}(u_{m_l}, u_{n_l}) \leq m_{2\eta}(u_{m_l}, u_{m_{l+1}}) + m_\eta(u_{m_{l+1}}, u_{n_{l+1}}) + m_\eta(u_{n_{l+1}}, u_{n_l}).$$

In (12), we take the limit on both sides so that we gain $\lim_{l \rightarrow \infty} m_\eta(u_{m_{l+1}}, u_{n_{l+1}}) \geq \epsilon$. Furthermore, we have

$$(13) \quad m_\eta(u_{m_l}, u_{n_l}) \leq w_{\eta/2}(u_{m_l}, u_{n_{l-1}}) + w_{\eta/2}(u_{n_{l-1}}, u_{n_l}),$$

which gives $\lim_{l \rightarrow \infty} m_\eta(u_{m_l}, u_{n_l}) \leq \epsilon$. In a similar way

$$(14) \quad m_\eta(u_{m_l}, u_{n_{l+1}}) \leq w_{\eta/2}(u_{m_l}, u_{n_{l-1}}) + w_{\eta/4}(u_{n_{l-1}}, u_{n_l}) + w_{\eta/4}(u_{n_l}, u_{n_{l+1}}).$$

We reach $\lim_{l \rightarrow \infty} m_\eta(u_{m_l}, u_{n_{l+1}}) \leq \epsilon$ as $l \rightarrow \infty$. Putting $x_1 = u_{n_l}$ and $x_2 = u_{m_l}$ in (7) yields

$$(15) \quad \frac{1}{2}m_\eta(u_{n_l}, Gu_{n_l}) = \frac{1}{2}m_\eta(u_{n_l}, u_{n_{l+1}}) \leq m_\eta(u_{n_l}, u_{m_l}).$$

Inequality (15) implies that

$$(16) \quad \begin{aligned} \psi(m_\eta(Gu_{n_l}, Gu_{m_l})) &= \psi(m_\eta(u_{n_{l+1}}, Gu_{m_{l+1}})) \\ &\leq \beta \left(\max \left\{ m_\eta(u_{n_l}, u_{m_l}), \frac{m_\eta(u_{n_l}, Gu_{n_l})(1 + m_\eta(u_{m_l}, Gu_{m_l}))}{1 + m_\eta(u_{n_l}, u_{m_l})}, \right. \right. \\ &\quad \left. \left. \frac{m_\eta(u_{m_l}, Gu_{n_l})(1 + m_\eta(u_{m_l}, Gu_{n_l}))}{1 + m_\eta(u_{n_l}, u_{m_l})} \right\} \right). \end{aligned}$$

We attain

$$(17) \quad \psi(\epsilon) \leq \lim_{l \rightarrow \infty} \psi(m_\eta(u_{n_{l+1}}, u_{m_{l+1}})) \leq \beta \left(\max \left\{ \epsilon, \frac{\epsilon}{1 + \epsilon}(1 + \epsilon) \right\} \right) = \beta(\epsilon)$$

as $l \rightarrow \infty$. This is a contradiction, which means that $\{u_n\}$ is m -Cauchy on Ω_m . As Ω_m is m -complete, there exists a $u \in \Omega_m$ such that $\{u_n\}$ is m -convergent to u , i.e., $\lim_{n \rightarrow \infty} m_\eta(u_n, u) = 0$. Since m is strict, the limit is unique. Now, we demonstrate that $Gu = u$. Let us put $\tau_0 = u_n$ and $\tau_1 = u$ in (7). Then we get

$$(18) \quad \frac{1}{2}m_\eta(u_n, Gu_n) = \frac{1}{2}m_\eta(u_n, u_{n+1}) \leq m_\eta(u_{n_l}, u).$$

This implies that

$$(19) \quad \begin{aligned} \psi(m_\eta(Gu_n, Gu)) &= \psi(m_\eta(u_{n+1}, Gu)) \\ &\leq \beta \left(\max \left\{ m_\eta(u_n, u), \frac{m_\eta(u_n, Gu_n)(1 + m_\eta(u, Gu))}{1 + m_\eta(u_n, u)}, \right. \right. \\ &\quad \left. \left. \frac{m_\eta(u, Gu_n)(1 + m_\eta(u, Gu_n))}{1 + m_\eta(u_n, u)} \right\} \right) \\ &\leq \beta \left(\max \left\{ m_\eta(u_n, u), \frac{m_\eta(u_n, u_{n+1})(1 + m_\eta(u, Gu))}{1 + m_\eta(u_n, u)}, \right. \right. \\ &\quad \left. \left. \frac{m_\eta(u, u_{n+1})(1 + m_\eta(u, u_{n+1}))}{1 + m_\eta(u_n, u)} \right\} \right). \end{aligned}$$

From (19), $\psi(m_\eta(u, Gu)) \leq \beta(0) < \psi(0)$ is gained as $n \rightarrow \infty$.

$\psi(m_\eta(u, Gu)) \leq 0 \implies m_\eta(u, Gu) = 0$ we deduce $Gu = u$ as m is strict.

It remains to be shown that the fixed point of G is unique. On the contrary, assume that G has two fixed points $u, v \in \Omega$ such that $Gu = u$ and $Gv = v$ with $u \neq v$. There are two cases to be considered.

For the first one, we assume that u and v are comparable with $u \preceq v$. Putting $\tau_0 = u$ and $\tau_1 = v$ in (7) yields

$$(20) \quad 0 = \frac{1}{2}m_\eta(u, Gu) = \frac{1}{2}m_\eta(u, u) \leq m_\eta(u, v).$$

It implies that

$$(21) \quad \begin{aligned} \psi(m_\eta(Gu, Gv)) &= \psi(m_\eta(u, v)) \leq \beta \left(\max \left\{ m_\eta(u, v), \right. \right. \\ &\quad \left. \left. \frac{m_\eta(u, Gu)(1 + m_\eta(v, Gv))}{1 + m_\eta(u, v)}, \frac{m_\eta(v, Gu)(1 + m_\eta(v, Gu))}{1 + m_\eta(u, v)} \right\} \right), \\ &\leq \beta \left(\max \left\{ m_\eta(u, v), \frac{m_\eta(u, u)(1 + m_\eta(v, v))}{1 + m_\eta(u, v)}, \frac{m_\eta(v, u)(1 + m_\eta(v, u))}{1 + m_\eta(u, v)} \right\} \right) \\ &\leq \beta(m_\eta(u, v)). \end{aligned}$$

Since, for all $\tau > 0$, $\psi(\tau) > \beta(\tau)$, then $m_\eta(u, v) = 0$. Hence, we gain $u = v$.

For the second one, we assume that u and v are not comparable. From the theorem hypothesis, there exists $w \in \Omega$, comparable to u and v . That is, $u \preceq w$ and $v \preceq w$. Now, we define a sequence w_n as $w_0 = w$ and $w_{n+1} = Gw_n$ for all $n \in N$. We first consider $u \preceq w$. Because G is nondecreasing, $u = Gu \preceq Gw = w_1$. Moreover, $u = Gu \preceq Gw_{n+1} = w_n$ for all $n \in N$.

Assume that $u = w_{n^*}$ for some $n^* \in N$. Then, $u = Gu \preceq Gw_{n^*} = w_{n^*}$, and so $u = G^p = w_p$ for all $p \geq n^*$. This gives that w_n is convergent to u , that is, $\lim_{n \rightarrow \infty} m_\eta(w_n, u) = 0$. Hence, suppose that $u \neq w_n$ for all $n \in N$. Then we can write

$$(22) \quad 0 = \frac{1}{2}m_\eta(u, Gu) \leq m_\eta(u, w_n) \implies$$

$$\psi(m_\eta(Gu, Gw_n)) \leq \beta \left(\max \left\{ m_\eta(u, w_n), \frac{m_\eta(u, Gu)(1 + m_\eta(w_n, Gw_n))}{1 + m_\eta(u, w_n)}, \right. \right.$$

$$\left. \left. \frac{m_\eta(w_n, Gu)(1 + m_\eta(w_n, Gu))}{1 + m_\eta(u, w_n)} \right\} \right).$$

Then we achieve

$$(23) \quad \psi(m_\eta(u, w_{n+1})) \leq \beta(m_\eta(u, w_n)).$$

Using the properties of ψ and β , we find $m_\eta(u, w_{n+1}) \leq m_\eta(u, w_n)$. As it is seen that $m_\eta(u, w_n)$ is decreasing and positive sequence. Hereby it is convergent, say $\lim_{n \rightarrow \infty} m_\eta(w_n, u) = L \geq 0$. Let us take the limit of both sides in (23).

Then, we have $\psi(L) \leq \beta(L)$, which is a contradiction. Hence, $L = 0$, that is, $\lim_{n \rightarrow \infty} m_\eta(w_n, u) = 0$.

In a similar way, one can reach that $\lim_{n \rightarrow \infty} m_\eta(w_n, v) = 0$. Consequently, we deduce $u = v$. This completes the proof. $\square$

The following theorem can be proved similarly with Theorem 12.

Theorem 13. *Let an MM m be strict on Ω and the triple $(\Omega_m, m, \preceq)$ be an m -complete partially ordered modular space w.r.t. ' $\preceq$ '. Suppose that $G: \Omega_m \rightarrow \Omega_m$ is satisfying the generalized C -condition of type II with $\psi(\tau) > \beta(\tau)$ for all $\tau > 0$. Assume that there exists $w \in \Omega$, comparable to u and v for any $h, g \in \Omega$. In addition, if a point $u_0 \in \Omega$ holds $u_0 \preceq Gu_0$, then G has a unique fixed point.*

Now, we hold forth two examples to support our results gained in the above theorems.

Example 14. Let us consider an MM m on $\Omega = [0, 1]$. If we take the strict MM $m_\eta(a, b) = d(a, b)$ where $d(a, b) = \max_{a, b \in \Omega} |a - b|$, then the MMS Ω_m given in (6) becomes m -complete induced by m . Moreover, $(\Omega_m, m, \preceq)$ is a partially ordered m -complete MMS w.r.t. ' $\preceq$ '. We consider the function $Gh = (\ln(h^2 + 10) + h)/10$ defined on $[0, 1]$. It is easy to see that $G: \Omega_m \rightarrow \Omega_m$.

We first show that the given function G satisfies the generalized C -condition of type I. If we choose the functions $\psi(h) = e^h - 1$ and $\beta(h) = h^2$, then $\psi(h) > \beta(h)$ holds for all $h > 0$. It is clear that $\psi \in \Psi$ and $\beta: [0, \infty) \rightarrow [0, \infty)$ is continuous.

Putting $\tau_0 = h$ and $\tau_1 = g$ in (7) yields

$$\begin{aligned} \frac{1}{2}m_\eta(h, Gh) &= \frac{1}{2}d(h, Gh) = \frac{1}{2}d\left(h, \frac{1}{10}(\ln(h^2 + 10) + h)\right) \\ (24) \qquad &= \frac{1}{2} \max_{h \in \Omega} \left| h - \frac{1}{10}(\ln(h^2 + 10) + h) \right| = 0.1151. \end{aligned}$$

Moreover, we find $m_\eta(h, g) = d(h, g) = \max_{h, g \in \Omega} |h - g| = 1$. Hereby, we get $\frac{1}{2}m_\eta(h, Gh) \leq m_\eta(h, g)$ for all $\eta > 0$. Moreover, we have

$$\begin{aligned} m_\eta(Gh, Gg) &= d(Gh, Gg) \\ (25) \qquad &= \max_{h, g \in \Omega} \left| \frac{1}{10}(\ln(h^2 + 10) + h) - \frac{1}{10}(\ln(g^2 + 10) + g) \right| \\ &\leq \frac{1}{10} \max_{h, g \in \Omega} \left\{ \left| \ln \frac{h^2 + 10}{g^2 + 10} \right| + |h - g| \right\} = 0.1095. \end{aligned}$$

Since $\psi(h) = e^h$, we gain $\psi(m_\eta(Gh, Gg)) \leq e^{0.1095} - 1 = 0.116$.

Now, let us calculate the second part. Since $d(h, g) = 1$ and $d(h, Gh) = d(g, Gh) = d(g, Gg) = 0.66$, we find the right hand side as $\beta(1)$. Since $\beta(h) = h^2$, we get $\beta(1) = 1$. Hereby, we have $\psi(0.1095) = 0.116 \leq \beta(1) = 1$.

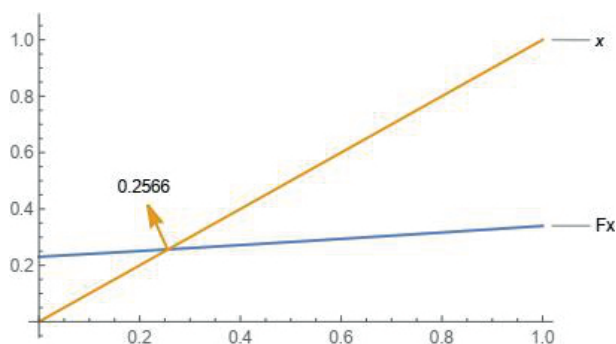


Fig. 1. The intersection of Fx given in Example 14 and $y = x$

Since $\Omega = [0, 1] \subset \mathbb{R}$, we can find an element $w \in [0, 1]$, comparable to h and g for all $h, g \in [0, 1]$. Furthermore, we have $h_0 = 0.2 \in \Omega = [0, 1]$ such that $h_0 = 0.2 \leq Gh_0 = 0.2507$.

Consequently, all hypotheses of Theorem 12 are held. Hence, G has a unique fixed point.

As it is seen in Fig. 1, the unique fixed point of G is $0.2566 \in \Omega = [0, 1]$, i.e., $G(0.2566) = 0.2566$.

Example 15. We take the same set $\Omega = [0, 1]$ given in Example 14 with the MM $m_\eta(a, b) = d(a, b)/\eta$ where $d(a, b) = \max_{a, b \in \Omega} |a - b|$. Hereby, $(\Omega_m, m, \leq)$ is partially ordered m -complete MMS w.r.t. ' $\leq$ '. We consider the function $Gh = \frac{1}{4}(\sin h + h^2 + 1)$ such that $F: \Omega_m \rightarrow \Omega_m$. Firstly, we state that the function yields the generalized C -condition of type II. Indeed, if we choose $\psi(h) = h$ and $\beta(h) = h/2$ satisfying the condition $\psi(h) > \beta(h)$ for all $h > 0$, then for $\tau_0 = h$ and $\tau_1 = g$ we obtain

$$\begin{aligned}
 \frac{1}{2}m_\eta(h, Gh) &= \frac{1}{2\eta}d(h, Gh) = \frac{1}{2\eta}d\left(h, \frac{1}{4}(\sin h + h^2 + 1)\right) \\
 (26) \quad &= \frac{1}{2\eta} \max_{h \in \Omega} \left| h - \frac{1}{4}(\sin h + h^2 + 1) \right| = \frac{0.1448}{\eta} \\
 &\leq m_\eta(h, g) = \frac{d(h, g)}{\eta} = \frac{1}{\eta} \max_{h, g \in \Omega} |h - g| = \frac{1}{\eta}.
 \end{aligned}$$

It is easy to find that $m_\eta(Gh, Gg) \leq 0.4604/\eta$. For the right part we have $\beta(1/\eta)$. Since $\psi(h) = h$ and $\beta(h) = h/2$, we obtain $\psi(m_\eta(Gh, Gg)) \leq 0.4604/\eta \leq \beta(1/\eta) = 0.5/\eta$.

As $\Omega = [0, 1] \subset \mathbb{R}$, we can find an element q in $[0, 1]$, comparable to h and g for all $h, g \in [0, 1]$. Moreover, there exists $h_0 = 0.1 \in [0, 1]$ such that $h_0 = 0.1 \leq Fh_0 = F(0, 1) = 0.2775$.

Therefore, we have all the conditions of Theorem 13 yielded, so G has a unique fixed point $h = 0.378$ in $\Omega = [0, 1]$ as shown in Fig. 2.

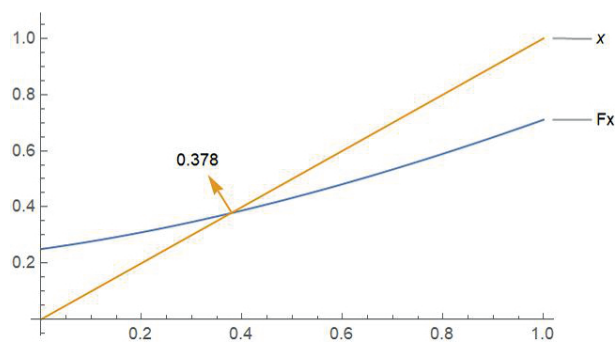


Fig. 2. The intersection of Fx given in Example 15 and $y = x$

4. Application. In this section, we investigate the existence and uniqueness of solution for the integral equation given as

$$(27) \quad h(t) = H(t) + \int_0^T M(t, s)Y(s, h(s)) ds, \quad t \in [0, T],$$

where the function $Y: [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is continuous and nondecreasing, $M: [0, T] \times [0, T] \rightarrow \mathbb{R}^+$ and $H: [0, T] \rightarrow \mathbb{R}$ are continuous functions.

We study this equation on the set of real valued continuous functions on $[0, T]$, i.e., $\Omega = C([0, T])$. We consider the following *MMS* around h_0

$$(28) \quad \Omega_m = \Omega_m(h_0) = \{h \in \Omega = C([0, T]): m_\eta(h, h_0) \rightarrow 0 \text{ as } \eta \rightarrow \infty\}.$$

Now, let us take the strict *MM* $m_\eta(a, b) = d(a, b)/\eta$ where $d(h, g) = \max_{t \in [0, T]} |h(t) - g(t)|$. Then the *MMS* Ω_m defined in (28) becomes *m*-complete. Furthermore, we can equip this space with the partial order defined as $h, g \in C([0, T])$ such that $h \leq g \implies h(t) \leq g(t)$ for all $t \in [0, T]$ so that Ω_m becomes partially ordered *m*-complete *MMS* w.r.t. $\leq$.

Now, let us define $G: C([0, T]) \rightarrow C([0, T])$ as

$$(29) \quad (Gh)(t) = H(t) + \int_0^T M(t, s)Y(s, h(s)) ds, \quad t \in [0, T].$$

It is clear that if $h(t)$ is a fixed point of G , then $h \in C([0, T])$ is the solution of the integral equation (27). For this purpose, we must show that all hypotheses in Theorem 3 are held.

Theorem 16. Suppose that the functions $M(t, s)$ and $Y(s, h)$ satisfy the following inequality

$$(30) \quad \max_{t \in [0, T]} \int_0^T |M(t, s)| ds \leq \frac{\eta}{T}$$

and for all $h, g \in C([0, T])$

$$(31) \quad |Y(s, h(s)) - Y(s, g(s))| \leq \sqrt[4]{\ln \left[\left(\frac{|h(s) - g(s)|}{\eta} \right)^5 + 1 \right]}.$$

Then the integral equation (27) has a unique solution.

Proof. We may take $h, g \in C([0, T])$ such that $h \leq g$. Since $Y(s, h)$ is nondecreasing, it is easy to see that $Gh \leq Gg$ which gives that F is nondecreasing.

Moreover, we have

$$(32) \quad g(t) = H(t) + \int_0^T M(t, s)Y(s, g(s)) ds \geq H(t) + \int_0^T M(t, s)Y(s, h(s)) ds = Gh(t).$$

From the above, we can write

$$(33) \quad \begin{aligned} m_\eta(h, g) &= \frac{d(h, g)}{\eta} = \frac{1}{\eta} \max_{t \in [0, T]} |g(t) - h(t)| \geq \frac{1}{\eta} \max_{t \in [0, T]} |Gh(t) - h(t)| \\ &= \frac{1}{\eta} d(Gh, h) = m_\eta(Gh, h). \end{aligned}$$

Hence, we acquire $\frac{1}{2}m_\eta(Gh, h) \leq m_\eta(Gh, h) \leq m_\eta(h, g)$. We will show that this implies $\psi(m_\eta(Gh, Gg)) \leq \beta(m_\eta(h, g))$.

First, we estimate $m_\eta(Gh, Gg)$ as

$$(34) \quad \begin{aligned} m_\eta(Gh, Gg) &= \frac{d(Gh, Gg)}{\eta} = \frac{1}{\eta} \max_{t \in [0, T]} |Gh(t) - Gg(t)| \\ &\leq \frac{1}{\eta} \max_{t \in [0, T]} \int_0^T |M(t, s)| |Y(s, h(s)) - Y(s, g(s))| ds \\ &\leq \frac{1}{\eta} \max_{t \in [0, T]} \int_0^T |M(t, s)| \sqrt[4]{\ln \left[\left(\frac{|h(s) - g(s)|}{\eta} \right)^5 + 1 \right]} ds \\ &\leq \frac{1}{T} \int_0^T \sqrt[4]{\ln \left[\left(\frac{|h(s) - g(s)|}{\eta} \right)^5 + 1 \right]} ds. \end{aligned}$$

As a result, we gain

$$(35) \quad m_\eta(Gh, Gg) \leq \sqrt[4]{\ln(m_\eta(h(s), g(s))^5 + 1)},$$

or, equivalently,

$$(36) \quad m_\eta(Gh, Gg)^4 \leq \ln(m_\eta(h(s), g(s))^5 + 1).$$

Moreover, we attain

$$(37) \quad \psi(m_\eta(Gh, Gg)) \leq \beta(m_\eta(h(s), g(s))),$$

where $\psi(h) = h^4$ and $\beta(h) = \ln(h^5 + 1)$. Obviously, $\psi \in \Psi$ and $\beta: [0, \infty] \rightarrow [0, \infty]$ is continuous and the condition $\beta(h) < \psi(h)$ holds for all $h > 0$.

We, therefore, have all hypotheses in Theorem 13 fulfilled; that is, G has a unique fixed point. Consequently, $h \in C([0, T])$ is the unique solution of (27). $\square$

5. Conclusion. We consider two types of the generalized C -condition in MMS . For such types of conditions, we give the fixed point theorems and their proofs. By using the results obtained here, we demonstrate that the given mappings in Examples 13 and 14 have unique fixed points in the mentioned $POMMS$. The fixed points of the mappings are illustrated in xy -plane by intersecting the graphs of the given mappings and the line $x = y$. Furthermore, we provide the existence of a solution for the given integral equation in the considered $POMMS$. In addition to this, it is stated that the solution is unique. The conclusions found here carry the fixed point results for the generalized C -conditions attained in [20] to $POMMS$.

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