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ON THE DIOPHANTINE EQUATION $L_m^2 + L_n^2 = 2^a$

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Abstract

This study researches numbers that are powers of two and can be represented as the sum of the squares of any two Lucas numbers. We apply Baker's theory of linear forms in logarithms of algebraic numbers, combined with a variation of the Baker–Davenport reduction method, to solve the Diophantine equation $L_m^2 + L_n^2 = 2^a$, where m, n and a are positive integers, as presented in this study.

Key words: Matveev theorem, Lucas number, Diophantine equation, linear forms in logarithms, Dujella–Pethö reduction lemma

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1. Introduction and motivation. Let $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ be the Fibonacci and Lucas sequences given by the recursive formulas $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, with the initial terms $F_0 = 0$, $L_0 = 2$, and $F_1 = L_1 = 1$, respectively. Also, Fibonacci and Lucas sequences can be generated with the Binet formulas in the following way:

$$F_n = \frac{\gamma^n - \delta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \gamma^n + \delta^n, \tag{1.1}$$

for all integers $n \geq 0$, where $\gamma = \frac{1+\sqrt{5}}{2}$ and $\delta = \frac{1-\sqrt{5}}{2}$, which are the roots of the characteristic equation $x^2 - x - 1 = 0$. Furthermore, it is readily apparent that $\gamma + \delta = 1$ and $\gamma\delta = -1$. Scientists have investigated Fibonacci and Lucas numbers

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in various fields of mathematics and geometry. For more detailed examples of Fibonacci and Lucas sequences, one can refer to [1].

In recent years, the study of several exponential Diophantine equations involving Fibonacci and Lucas numbers has increased. The case in which the sums or differences of Fibonacci or Lucas numbers equal a Fibonacci number, a Lucas number, or a perfect power has been investigated. Bravo and Luca [2] examined the Diophantine equation $F_m + F_n = 2^a$ and found all its solutions. In addition, Bravo and Bravo [3] investigated and found all solutions to the Diophantine equation $F_m + F_n + F_l = 2^a$. Similarly, Bravo and Luca investigated the same problem for Lucas numbers in [4]. In other words, they researched and found the solutions of the Diophantine equation $L_m + L_n = 2^a$. Şiar and Keskin [5] determined that perfect powers can be expressed as the sum or difference of two Lucas numbers.

Many studies have investigated similar relationships among Fibonacci, Lucas, and other integer sequences. Several example papers on this topic can be found; readers interested in further research can refer to the papers listed in [7–12].

The problem of identifying all perfect powers within the Fibonacci and Lucas sequences remained unsolved for a considerable period until Bugeaud et al. [13] resolved it in 2006. That is, the only perfect powers in the Fibonacci sequence are 0, 1, 8, and 144, while in the Lucas sequence, they are 1 and 4.

Based on the findings from the literature review, it has yet to be investigated whether the sum of the squares of any two Lucas numbers equals a power of two. Therefore, in this study, we determine all the solutions of the Diophantine equation

$$L_m^2 + L_n^2 = 2^a \tag{1.2}$$

where $a \geq 1$ and $1 \leqslant m \leqslant n$.

2. Preliminaries. This section of the paper provides fundamental definitions, results, and notations from algebraic number theory.

It is possible to see the following lemma in many scientific articles related to the subject.

Lemma 2.1. *The inequality*

$$\gamma^{n-1} \leqslant L_n \leqslant 2\gamma^n \tag{2.1}$$

holds for all $n \geq 0$.

Proof. The proof can be proved by induction on n. □

Let ι be an algebraic number of degree z and

$$c_0 x^z + c_1 x^{z-1} + \cdots + c_z = \sum_{j=0}^{z} c_j x^{z-j}$$

be its minimal polynomial in $\mathbb{Z}[x]$, where c_j's are relatively prime integers with $c_0 > 0$. The logarithmic height of ι is denoted by $h(\iota)$ and defined by

$$h(\iota) = z^{-1}\left(\log c_0 + \sum_{i=1}^{z} \log\left(\max\left\{\left|\iota^{(i)}\right|, 1\right\}\right)\right), \tag{2.2}$$

where $\iota^{(i)}$'s are the conjugates of ι.

There are also numerous properties related to logarithmic height mentioned in the references. These properties are as follows:

$$h(\iota_1 + \iota_2) \leq h(\iota_1) + h(\iota_2) + \log 2, \tag{2.3}$$

$$h\left(\iota_1 \iota_2^{\pm 1}\right) \leq h(\iota_1) + h(\iota_2), \tag{2.4}$$

$$h(\iota^r) = |r| h(\iota). \tag{2.5}$$

Let $\iota_1, \iota_2, \ldots, \iota_r$ be non-zero real algebraic numbers in a number field $\mathbb{F}$ of degree $d_{\mathbb{F}}$, and let $k_1, k_2, \ldots, k_r$ be non-zero rational numbers. Also

$$\Lambda = \iota_1^{k_1} \iota_2^{k_2} \ldots \iota_r^{k_r} - 1 \quad \text{and} \quad B \geq \max\{|k_1|, |k_2|, \ldots, |k_r|\}.$$

Let $A_1, A_2, \ldots, A_r$ be the positive real numbers such that

$$A_j \geq \max\{d_{\mathbb{F}} h(\iota_j), |\log \iota_j|, 0.16\} \quad \text{for all} \quad j = 1, 2, \ldots, r. \tag{2.6}$$

Based on the notations mentioned above, an important theorem established by Matveev [14], will be presented as follows:

Theorem 2.1 (Matveev [14]). *If $\Lambda \neq 0$ and $\mathbb{F}$ is a real algebraic number field of degree $d_{\mathbb{F}}$, then*

$$\log(|\Lambda|) > -1.4 \times 30^{r+3} \times r^{4.5} \times d_{\mathbb{F}}^2 \times (1 + \log d_{\mathbb{F}}) \times (1 + \log B) \times A_1 \times A_2 \times \cdots \times A_r.$$

To reduce the bounds from applying Theorem 2.1, the following lemma was developed by Dujella and Pethö [15].

Lemma 2.2 (Dujella and Pethö [15]). *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let W, V, μ be real numbers with $W > 0$ and $V > 1$. Let $\varepsilon := \|\mu q\| - M\|\tau q\|$, where $\|\cdot\|$ is the distance from the nearest integer. If $\varepsilon > 0$, then there is no integer solution (a, n, l) of inequality*

$$0 < a\tau - n + \mu < WV^{-l}$$

with

$$a \leq M \quad \text{and} \quad l \geq \frac{\log(Wq/\varepsilon)}{\log V}.$$

The next theorem, credited to Legendre, will be applied in some parts of our investigation.

Theorem 2.2 (Legendre [16]). *Let τ be a reel number, p, q be integers and let $\tau = [a_0, a_1, \dots]$. If*

$$\left|\frac{p}{q} - \tau\right| < \frac{1}{2q^2},$$

then p/q is a convergent of continued fractions of τ. In addition, let M and N be non-negative integers with $q_N > M$. Put $x := \max\{a_i\}$ for $i = 0, 1, 2, \dots, N$, then

$$\frac{1}{(x+2)q^2} < \left|\frac{p}{q} - \tau\right|.$$

3. Main results. The fundamental result of the paper is given below.

Theorem 3.1. *Let m, n, and a be positive integers. Then, equation* (1.2) *is satisfied only for the triples of*

$$(m, n, a) \in \{(1, 1, 1), (3, 3, 5)\}. \tag{3.1}$$

Proof. Assume that equation (1.2) holds. Firstly, if $n = m$, then equation (1.2) becomes $L_n^2 = 2^{a-1}$. If [13, Theorem 1] is taken into consideration, it can be seen that the solutions to this equation are $(n, a) \in \{(1, 1), (3, 5)\}$, which are the solutions stated in (3.1). Therefore, for the rest of the paper, we assume that $m < n$.

If m and n are two consecutive integers, we can consider the equation $L_n^2 + L_{n+1}^2 = L_{2n} + L_{2n+2}$ given in [1]. If $L_m^2 + L_n^2 = 2^a = L_{2n} + L_{2n+2}$, we are looking for sums of two Lucas numbers that are powers of two, as shown by Bravo and Luca in [4]. However, there exist no pairs (n, a) such that $1 \le m < n$ and $a \ge 1$ satisfying this condition.

If $n \le 300$, a brute-force search using Maple© for $1 \le m < n \le 300$, after a process that took approximately ten hours, yields the solutions $(m, n, a) \in \{(1, 1, 1), (3, 3, 5)\}$. Henceforth, we will consider $n > 300$ for the remainder of the paper.

Now we get a relation between n and a. Considering Lemma 2.1 and equation (1.2), we can write

$$2^a = L_m^2 + L_n^2 \le 4\gamma^{2m} + 4\gamma^{2n} < 4\gamma^{2n} + 4\gamma^{2n} = 8 \cdot \gamma^{2n} < 2^{2n+3} \tag{3.2}$$

hence $a < 2n + 3$. This relation between a and n will be used many times in different parts of this paper.

Applying the Binet's formulas in equation (1.1) to equation (1.2) yields

$$L_m^2 + L_n^2 = 2^a \Rightarrow (\gamma^m + \delta^m)^2 + (\gamma^n + \delta^n)^2 = 2^a \tag{3.3}$$

and from this, we get

$$2^a - \gamma^{2n} = \gamma^{2m} + 2((-1)^m + (-1)^n) + \delta^{2m} + \delta^{2n}.$$

Dividing both sides of the last equation by γ^{2n}, taking absolute values, and taking into account that $m < n$, we get

$$\begin{aligned}|2^a \cdot \gamma^{-2n} - 1| &= \left| \frac{1}{\gamma^{2n-2m}} + \frac{2((-1)^m + (-1)^n)}{\gamma^{2n}} + \frac{\delta^{2m}}{\gamma^{2n}} + \frac{\delta^{2n}}{\gamma^{2n}} \right| \\ &\leq \frac{1}{\gamma^{2n-2m}} + \frac{2|(-1)^m + (-1)^n|}{\gamma^{2n}} + \frac{|\delta|^{2n}}{\gamma^{2n}} + \frac{|\delta|^{2m}}{\gamma^{2n}} < \frac{7}{\gamma^{n-m}}.\end{aligned}$$

As a result, we have

$$|\Lambda_1| < \frac{7}{\gamma^{n-m}}, \quad \Lambda_1 := 2^a \cdot \gamma^{-2n} - 1. \tag{3.4}$$

According to Theorem 2.1, we get $r := 2$, $\iota_1 := 2$, $\iota_2 := \gamma$, $k_1 := a$, $k_2 := -2n$. Because of $\iota_1, \iota_2 \in \mathbb{Q}\left(\sqrt{5}\right)$, we should consider $\mathbb{F} := \mathbb{Q}\left(\sqrt{5}\right)$ of degree $d_{\mathbb{F}} := 2$. It is clear that $\Lambda_1 \neq 0$. Indeed, if $\Lambda_1 = 0$, we get $2^a = \gamma^{2n}$, which is impossible because $2^a \in \mathbb{Z}$, but γ^{2n} is not. So, $\Lambda_1 \neq 0$. From equations (2.2) and (2.6), we can compute $h(\iota_1) = \log 2$, $h(\iota_2) = \frac{1}{2}\log\gamma$. Then, we can take $A_1 := 2\log 2$, and $A_2 := \log\gamma$, since $A_i \geq d_{\mathbb{T}} h(\iota_i)$, $i = 1, 2$. Besides, for $B := 2n + 3$, $B \geq \max\{a, |-2n|\}$, since $a < 2n + 3$. As a result, based on Theorem 2.1, we get

$$\log(\Lambda_1) > -1.4 \times 30^5 \times 2^{4.5} \times 2^2 \times (1 + \log 2) \times (1 + \log(2n + 3)) \times 2\log 2 \times \log\gamma$$

and with certain mathematical simplifications of the above inequality, we obtain

$$\log(\Lambda_1) > -1.75 \times 10^{10} \times \log n, \tag{3.5}$$

where we used the fact that $1 + \log(2n + 3) < 5\log n$ for $n \geq 2$. Further from inequality (3.4), we have

$$\log(\Lambda_1) < \log 7 - (n - m)\log\gamma. \tag{3.6}$$

Considering the inequalities (3.5) and (3.6) together, we get that

$$(n - m)\log\gamma < 1.76 \times 10^{10} \log n. \tag{3.7}$$

By the way, if rearranging the equation (1.2) as

$$\begin{aligned}L_m^2 + L_n^2 = 2^a &\Rightarrow (\gamma^m + \delta^m)^2 + (\gamma^n + \delta^n)^2 = 2^a \\ &\Rightarrow \gamma^{2n} + \gamma^{2m} - 2^a = -2(-1)^m - 2(-1)^n - \delta^{2m} - \delta^{2n}.\end{aligned}$$

Taking absolute values after dividing both sides of the last equation by $\gamma^{2n}(1 + \gamma^{2m-2n})$ and taking into account that $m < n$, we get

$$\left| 2^a \cdot \gamma^{-2n} \left(1 + \gamma^{2m-2n}\right)^{-1} - 1 \right| < \frac{7}{\gamma^n}$$

and

$$|\Lambda_2| < \frac{7}{\gamma^n}, \quad \Lambda_2 := 2^a \cdot \gamma^{-2n} \left(1 + \gamma^{2m-2n}\right)^{-1} - 1. \tag{3.8}$$

To apply Matveev theorem into equation (3.8), we can consider that case where $r = 3$, $\iota_1 = 2$, $\iota_2 = \gamma$, $\iota_3 = \left(1 + \gamma^{2m-2n}\right)^{-1}$, $k_1 = a$, $k_2 = -2n$, and $k_3 = 1$. Since $\iota_1, \iota_2, \iota_3 \in \mathbb{Q}\left(\sqrt{5}\right)$ we can take $\mathbb{F} = \mathbb{Q}\left(\sqrt{5}\right)$ of degree $d_{\mathbb{F}} = 2$. As can be seen, since $\gamma^{2n} + \gamma^{2m} = 2^a$ is never satisfied, $\Lambda_2 \neq 0$. Besides, if we take $B = 2n + 3$, then $B \geq \max\{a, |-2n|, 1\}$, since $a < 2n + 3$. In this case, we can compute the following:

$$h(\iota_1) = \log 2, \quad h(\iota_2) = \frac{1}{2}\log\gamma, \quad A_1 = 2\log 2, \quad \text{and} \quad A_2 = \log\gamma.$$

From (2.3)–(2.6) we get

$$h(\iota_3) \leq \log 2 + (n - m)\log\gamma.$$

Therefore, we can take

$$A_3 = 2 + 2(n - m)\log\gamma = d_{\mathbb{F}}(1 + (n - m)log\gamma) \geq d_{\mathbb{F}} h(\iota_3).$$

In this case, according to Matveev's theorem, we can write

$$\log(\Lambda_2) > -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2) \times (1 + \log(2n + 3)) \times 2\log 2 \\ \times \log\gamma \times (2 + 2(n - m)\log\gamma)$$

and with certain mathematical simplifications of the above inequality, we obtain,

$$\log(\Lambda_2) > -6 \times 10^{12} \times \log n \times (1 + (n - m)\log\gamma), \tag{3.9}$$

where we used the fact that $1 + \log(2n + 3) < 5\log n$, for $n \geq 2$. From the right-hand side of the inequality (3.8) we get

$$\log\Lambda_2 < \log 7 - n\log\gamma. \tag{3.10}$$

Considering the inequalities (3.7), (3.9) and (3.10), we deduce that

$$n < 6.1 \times 10^{24}. \tag{3.11}$$

Thus, we can summarize the results mentioned above with a lemma as follows:

Lemma 3.1. *All the possible solutions of the equation* (1.2) *are over the ranges* $m < n$, $n > 300$, *and* $a < 2n + 3 < 1.5 \times 10^{25}$.

3.1. Reducing the bounds on n. As can be seen, we have determined a finite number of solutions to our problems, even though it has pretty extensive borders. The next step is to reduce the borders to a size that can be easily handled. We will use the Dujella–Pethö reduction lemma several times in this subsection to achieve this.

Firstly, we consider the notation

$$\Gamma_1 := a\log 2 - 2n\log\gamma. \tag{3.12}$$

Then, inequality (3.4) can be rewritten as

$$|\Lambda_1| = \left|e^{\Gamma_1} - 1\right| < \frac{7}{\gamma^{n-m}}. \tag{3.13}$$

Secondly, by using (1.1) and (1.2), we can write

$$\gamma^{2n} = L_n^2 - 2(-1)^n - \delta^{2n} \leq L_n^2 + 2 < L_n^2 + L_m^2 = 2^a$$

for $m > 1$. Hence, $1 < 2^a \cdot \gamma^{-2n}$ and so, $\Gamma_1 > 0$. Considering this inequality with (3.13), we obtain

$$0 < \Gamma_1 \leq e^{\Gamma_1} - 1 < \frac{7}{\gamma^{n-m}},$$

where we used the fact that $s \leq e^s - 1$ for all $s \in \mathbb{R}$. Substituting Γ_1 in the above inequality by its formula (3.12), we get

$$\Gamma_1 = a\log 2 - 2n\log\gamma < \frac{7}{\gamma^{n-m}}, \tag{3.14}$$

and dividing both sides of the above inequality by $2\log\gamma$, we have

$$0 < a\frac{\log 2}{2\log\gamma} - n < \frac{7}{\gamma^{n-m}\cdot 2\log\gamma} < \frac{8}{\gamma^{n-m}}. \tag{3.15}$$

Now, we use Legendre theorem, which is Theorem 2.2, to get a better bound on $n-m$. Indeed, $n-m < 134$. On the contrary, suppose that $n-m \geq 134$. Since

$$\gamma^{n-m} \geq \gamma^{134} > 10^{28} > 2.5\times 10^{26} > 16\cdot(1.5\times 10^{25}) > 16\cdot(2n+3) > 16\cdot a,$$

where $a < 2n+3 < 1.5\times 10^{25}$. From inequality (3.15) we have that

$$\left|\frac{\log 2}{2\log\gamma} - \frac{n}{a}\right| < \frac{8}{\gamma^{n-m}a} < \frac{8}{16a^2} = \frac{1}{2a^2}$$

which means that $\frac{n}{a}$ is a convergent of continued fractions of $\frac{\log 2}{2\log\gamma}$.

Let $\tau := \frac{\log 2}{2\log\gamma}$, $a < 2n+3 < M = 1.5\times 10^{25}$ by Lemma 3.1 and

$$[a_0, a_1, a_2, a_3, \dots] = [0, 1, 2, 1, 1, 2, 1, 6, \dots]$$

be the continued fraction expansion of τ. Also, let p_k/q_k denote its k-th convergent, where p_k and q_k are relatively prime integers. After a brief examination with Maple©, it becomes apparent that

$$3\,406\,276\,162\,917\,770\,819\,753\,599 = q_{55} < M = 1.5 \times 10^{25} < q_{56} = 37\,510\,821\,101\,450\,278\,173\,701\,164.$$

Moreover, $x := \max\{a_i\} = a_{19} = 67$ for $i = 1, 2, 3, \ldots, 56$. Therefore, from Theorem 2.2 we get that

$$\frac{1}{(67+2)a^2} < \left|\frac{\log 2}{2\log\gamma} - \frac{n}{a}\right| < \frac{8}{\gamma^{n-m}a}. \tag{3.16}$$

Thus, the last inequality implies that

$$a > \frac{\gamma^{n-m}}{8 \cdot 69} \geq \frac{\gamma^{134}}{8 \cdot 69} > 1.9 \times 10^{25}$$

which is a contradiction since $a < 1.5 \times 10^{25}$. Therefore, it follows that $n - m \leq 133$. When we insert this upper bound for $n - m$ into (3.9), the result is $n < 3.1 \times 10^{16}$.

Now we consider the notation

$$\Gamma_2 := a\log 2 - 2n\log\gamma + \log\left(\frac{1}{1+\gamma^{2m-2n}}\right) \tag{3.17}$$

and

$$|\Lambda_2| = \left|e^{\Gamma_2} - 1\right| < \frac{7}{\gamma^n}. \tag{3.18}$$

Since $\Lambda_2 \neq 0$, we observe that $\Gamma_2 \neq 0$. So, we differentiate the following cases. If $\Gamma_2 > 0$, then $e^{\Gamma_2} - 1 > 0$, hence from (3.18) and using the fact that $s \leq e^s - 1$ for all $s \in \mathbb{R}$, we get

$$0 < \Gamma_2 < e^{\Gamma_2} - 1 < \frac{7}{\gamma^n}.$$

Assume now that $\Gamma_2 < 0$. It is easy to see that $\frac{7}{\gamma^n} < \frac{1}{2}$ for all $n > 300$. So, from (3.18), we have that $\left|e^{\Gamma_2} - 1\right| < \frac{1}{2}$. Hence, $e^{|\Gamma_2|} < 2$. Because of $\Gamma_2 < 0$, we can deduce that

$$0 < |\Gamma_2| < e^{|\Gamma_2|} - 1 = e^{|\Gamma_2|}\left|e^{\Gamma_2} - 1\right| < \frac{14}{\gamma^n}.$$

In any case, we obtain that the inequality

$$0 < |\Gamma_2| < \frac{14}{\gamma^n}$$

holds for all $n > 300$. By replacing Γ_2 in the previous inequality with its formula and following the argument in (3.17) and dividing both sides of the inequality by $2\log\gamma$, we have

$$0 < \left| a\frac{\log 2}{2\log\gamma} - n + \frac{\log\left(1/\left(1+\gamma^{2m-2n}\right)\right)}{2\log\gamma} \right| < \frac{14}{\gamma^n \cdot 2\log\gamma} < \frac{15}{\gamma^n}.$$

From Lemma 3.1, we put $M := 1.5 \times 10^{25}$, $M > 2n+3 > a$. We also put

$$\tau := \frac{\log 2}{2\log\gamma}, \quad \mu := \frac{\log(1/(1+\gamma^{2m-2n}))}{2\log\gamma}, \quad W := 15, \quad V := \gamma \quad \text{and} \quad l := n.$$

Obviously τ is an irrational number and

$$\frac{p_{67}}{q_{67}} = \frac{364\,889\,102\,596\,710\,337\,962\,850\,590\,108}{506\,642\,617\,699\,397\,667\,695\,263\,997\,821}$$

is 67-th convergent of the continued fraction expansion of τ. We thus utilize Lemma 2.2 for all $n-m$ values in the range $[1,133]$. So, we get that

$$n \le \frac{\log(Wq_{67}/\varepsilon)}{\log V} < 272.173,$$

where $6M < q_{67}$ and $\varepsilon := \|\mu q_{67}\| - M\|\tau q_{67}\| > 0$, $\varepsilon > 10^{-26}$. If (m,n,a) is a possible solution of equation (1.2), then $n \le 272$. This statement is false because of our assumption that $n > 300$. □

Remark 1. The equation $\mu := \frac{\log\left(1/\left(1+\gamma^{2m-2n}\right)\right)}{2\log\gamma}$ has been examined for all values of $-2(n-m)$ in the range $1 \le n-m \le 133$ using Maple©. However, it has been observed that μ is not an integer for any value in this range. If μ were equal to an integer for any value in this range, these cases would be specially examined separately.

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