

ON GREATEST COMMON DIVISORS OF CONSECUTIVE SHIFTED FIBONACCI SEQUENCES

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Abstract

The shifted Fibonacci numbers which are formed by adding an integer a to the Fibonacci numbers firstly appeared in literature in 1973. It is well known that consecutive Fibonacci numbers are relatively prime. It has been a matter of curiosity among researchers that computing the greatest common divisor (gcd) of consecutive shifted Fibonacci numbers $F_n + a$ and $F_{n+1} + a$ for any a . In 1971, it was proven that consecutive shifted Fibonacci numbers are not always relatively prime and also that $\gcd(F_n + a, F_{n+1} + a)$ is unbounded for $a = \pm 1$. Recently, there has been an increasing interest in the greatest common divisors of shifted Fibonacci (Lucas) numbers, and in the studies conducted, they have been investigated for $a = \pm 1, \pm 2, \pm 3$. In this paper, the greatest common divisors are examined for consecutive Fibonacci and Lucas numbers shifted by $a = \pm 4$ and it is noticed that $\gcd(F_n \pm 4, F_{n+1} \pm 4)$ and $\gcd(L_n \pm 4, L_{n+1} \pm 4)$ are bounded.

Key words: Fibonacci numbers, Lucas numbers, shifted Fibonacci sequence, shifted Lucas sequence, greatest common divisor

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1. Introduction. The Fibonacci and Fibonacci-like sequences are special sequences that have garnered much interest from many researchers. They have been investigated by authors from different perspectives, see [1–8]. On the other hand, the Fibonacci sequence encounters in independent fields as well as mathematics in an interesting way. Recently, the Fibonacci sequence and the golden ratio related to the Fibonacci sequence have also become a popular topic for researchers in the fields of high energy physics, quantum mechanics, cryptography and coding. Fibonacci sequences play a very important role especially in coding theory, and in different forms are widely used in the creation of security coding.

The concept of the greatest common divisor (gcd) is used for various applications in number theory, especially in modular arithmetic, in encryption algorithms such as RSA (where messages are transmitted securely over the internet), and in simple applications such as simplifying fractions. This makes the greatest common divisor a fundamental concept for number theory, and algorithms are being developed to calculate the greatest common divisor efficiently. The greatest common divisor of two Fibonacci numbers is always a Fibonacci number, and so, consecutive Fibonacci (Lucas) numbers are relatively prime. For this reason, finding the greatest common divisors of special number sequences such as Fibonacci (Lucas) and examining the obtained results has become an interesting field of study.

For $n \geq 3$, define the generalized Fibonacci sequence $\{G_n\}$ by

$$(1.1) \quad G_n = G_{n-1} + G_{n-2}$$

with $G_1 = p$, $G_2 = q$. From [9], we know that

$$(1.2) \quad G_n = pF_{n-2} + qF_{n-1}.$$

If $p = q = 1$, then the generalized Fibonacci sequence G_n is the Fibonacci sequence F_n , and $p = 1$ and $q = 3$, G_n is the Lucas sequence L_n .

The shifted Fibonacci number which is formed by adding an integer a to a Fibonacci number firstly appeared in literature in 1973. There are some papers in the literature engaged with the greatest common divisor of the sequences which involves terms of the shifted Fibonacci numbers such as [10–15]. It is known that $\gcd(F_n + a, F_{n+1} + a) = 1$ for $a = 0$, i.e., successive members of the Fibonacci sequence are coprime. DUDLEY and TUCKER [10] first defined two sequences, $G_n = F_n + (-1)^n$ and $H_n = F_n - (-1)^n$ which they called altered Fibonacci numbers. The researchers calculated the greatest common divisors of some consecutive terms of these two sequences. They showed that $\gcd(G_{4n+k}, G_{4n+k+1})$ and $\gcd(H_{4n+k}, H_{4n+k+1})$ are not relatively prime for $k = 0, 2$. In summary, Dudley and Tucker calculated the greatest common divisors for $a = \pm 1$ and noticed that the greatest common divisors are equal to either a Fibonacci number or a Lucas number. Thus, they proved that the greatest common divisor for $a = \pm 1$ is unbounded. In 2011, CHEN [11] proved that the greatest common

divisor of consecutive shifted Fibonacci numbers is bounded unless ± 1 . He computed $\gcd(F_n + a, F_{n+1} + a)$ for $a = \pm 1$ and $a = \pm 2$ and $\gcd(L_n + a, L_{n+1} + a)$ for $a = 1$. The results showed that the greatest common divisor is bounded as claimed. RAHN and KREH [12] determined $\gcd(F_n + a, F_{n+1} + a)$ for $a = \pm 3$ and compared the results with those given by Chen [11]. KÖKEN [13] defined two Lucas sequences shifted by $\pm 1, \pm 3$, which he called altered Lucas sequences. He examined these sequences for some values of r , which he called r -consecutive greatest common divisor sequences. In 2016, SPILKER [14] proved that $\gcd(F_n + s, F_{n+1} + s)$ divides $s^2 + (-1)^n$. In 2020, CHEN and PAN [15], studied on shifted Horadam sequences and generalized $\gcd(F_{6n+3} + 1, F_{6n+6} + 1) = L_{3n+2}$.

In this study, the greatest common divisors are computed for consecutive Fibonacci and Lucas numbers shifted by $a = \pm 4$. Also, we noticed that $\gcd(F_n \pm 4, F_{n+1} \pm 4)$ and $\gcd(L_n \pm 4, L_{n+1} \pm 4)$ are bounded. Moreover, the results show that $\gcd(F_n \pm 4, F_{n+1} \pm 4)$ divide $(\pm 4)^2 + (-1)^n$ as claimed by Spilker [14].

In the next section, we will give some auxiliary lemmas before our main results.

2. Preliminaries. In this section, some properties related to divisibility and greatest common divisors that are necessary for the proof of the results will be given. Firstly, we have the following.

Lemma 2.1. *For any integers a, b and c ,*

$$(i) \gcd(a, bc) = \gcd(a, \gcd(a, b)c).$$

$$(ii) \gcd(a, b) = \gcd(a + bc, b).$$

With the following lemma we give the relationship between Fibonacci and Lucas numbers, their divisibility and greatest common divisor properties without proof.

Lemma 2.2 ([9]). *For all $m, n \in \mathbb{N}$,*

$$(i) F_{n-1} + F_{n+1} = L_n.$$

$$(ii) \gcd(F_n, F_{n+1}) = 1.$$

$$(iii) \gcd(F_m, F_n) = F_{\gcd(m, n)}.$$

$$(iv) L_n \equiv 2 \cdot 3^n \pmod{5}.$$

$$(v) n \equiv 2 \pmod{4} \Leftrightarrow L_n \equiv 0 \pmod{3}.$$

$$(vi) n \equiv 0 \pmod{m} \Leftrightarrow F_n \equiv 0 \pmod{F_m}.$$

We require the following lemma as the core of the proofs before determining the greatest common divisors.

Lemma 2.3 ([11]). *For every $m, k, a \in \mathbb{Z}$, the following holds*

$$(2.1) \quad \gcd(G_m + a, G_{m+1} + a) = \gcd(G_{m-2k} + aF_{2k-1}, G_{m-(2k+1)} - aF_{2k}).$$

From now on, we give some greatest common divisor values encountered in the proofs of the main results.

Lemma 2.4. *For each $n \in \mathbb{Z}$, it holds that*

$$(2.2) \quad \gcd(15, F_{2n} + 5F_{2n-1}) = \begin{cases} 3, & n \equiv 1 \pmod{2} \wedge n \not\equiv 0 \pmod{5}; \\ 5, & n \not\equiv 1 \pmod{2} \wedge n \equiv 0 \pmod{5}; \\ 15, & n \equiv 1 \pmod{2} \wedge n \equiv 0 \pmod{5}; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. We need to show that $3|F_{2n} + 5F_{2n-1}$ if and only if $n \equiv 1 \pmod{2}$ and $5|F_{2n} + 5F_{2n-1}$ if and only if $n \equiv 0 \pmod{5}$. By Lemma 2.2, we get

$$(2.3) \quad F_{2n} + 5F_{2n-1} = F_{2n+1} + F_{2n-1} + 3F_{2n-1} = L_{2n} + 3F_{2n-1}.$$

If $n \equiv 1 \pmod{2}$, then $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$ which implies that $2n \equiv 2 \pmod{4}$. Thus, from Lemma 2.2, $L_{2n} \equiv 0 \pmod{3}$. Hence, from Eq. (2.3), $F_{2n} + 5F_{2n-1} \equiv 0 \pmod{3}$. On the other hand, let $n \equiv 0 \pmod{5}$. By Lemma 2.2, we have

$$(2.4) \quad F_{2n} + 5F_{2n-1} \equiv 0 \pmod{5}.$$

Furthermore, the reverse is also true. Therefore,

$$(2.5) \quad \gcd(15, F_{2n} + 5F_{2n-1}) = \begin{cases} 3, & n \equiv 1 \pmod{2} \wedge n \not\equiv 0 \pmod{5}; \\ 5, & n \not\equiv 1 \pmod{2} \wedge n \equiv 0 \pmod{5}; \\ 15, & n \equiv 1 \pmod{2} \wedge n \equiv 0 \pmod{5}; \\ 1, & \text{otherwise.} \end{cases}$$

As a result, the proof is completed. □

Lemma 2.5. *For each $n \in \mathbb{Z}$, it holds that*

$$(2.6) \quad \gcd(15, 5F_{2n} - F_{2n-1}) = \begin{cases} 5, & \text{if } n \equiv 3 \pmod{5}; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. It suffices to show that $5|5F_{2n} - F_{2n-1}$ if and only if $n \equiv 3 \pmod{5}$ and also that $3 \nmid 5F_{2n} - F_{2n-1}$. By Lemma 2.2, $n \equiv 3 \pmod{5}$ if and only if $F_{2n-1} \equiv 0 \pmod{5}$ if and only if $5|5F_{2n} - F_{2n-1}$. On the other hand, from Lemma 2.2, $5F_{2n} - F_{2n-1} = 3F_{2n} + L_{2n-1}$. Moreover, it can be seen that $L_{2n-1} \not\equiv 0 \pmod{3}$ by Lemma 2.2. Then, $5F_{2n} - F_{2n-1} \not\equiv 0 \pmod{3}$. Hence, it is obvious that $\gcd(15, 5F_{2n} - F_{2n-1}) = 1$ for $n \not\equiv 3 \pmod{5}$ which completes the proof. □

3. Main results. In this section, we will give the greatest common divisors of consecutive Fibonacci and Lucas numbers shifted by ± 4 . We used the form $\gcd(G_{4n+k} + a, G_{4n+k+1} + a)$ to investigate the greatest common divisor, where G_n is either Fibonacci or Lucas sequences. We computed that for $a = \pm 4$, and showed that $\gcd(F_n \pm 4, F_{n+1} \pm 4)$ is a divisor of $(\pm 4)^2 + (-1)^n$.

Theorem 3.1. *Let n be any integer. Then we have*

1. $\gcd(F_{4n-1} + 4, F_{4n} + 4) = 1$.
2. $\gcd(F_{4n} + 4, F_{4n+1} + 4) = \begin{cases} 17, & \text{if } n \equiv 7 \pmod{9}; \\ 1, & \text{otherwise.} \end{cases}$
3. $\gcd(F_{4n+1} + 4, F_{4n+2} + 4) = \begin{cases} 3, & \text{if } n \equiv 1 \pmod{2} \wedge n \not\equiv 0 \pmod{5}; \\ 5, & \text{if } n \not\equiv 1 \pmod{2} \wedge n \equiv 0 \pmod{5}; \\ 15, & \text{if } n \equiv 1 \pmod{2} \wedge n \equiv 0 \pmod{5}; \\ 1, & \text{otherwise.} \end{cases}$
4. $\gcd(F_{4n+2} + 4, F_{4n+3} + 4) = 1$.

Theorem 3.2. *Let n be any integer. Then we have*

1. $\gcd(F_{4n-1} - 4, F_{4n} - 4) = \begin{cases} 5, & \text{if } n \equiv 3 \pmod{5}; \\ 1, & \text{otherwise.} \end{cases}$
2. $\gcd(F_{4n} - 4, F_{4n+1} - 4) = 1$.
3. $\gcd(F_{4n+1} - 4, F_{4n+2} - 4) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod{2}; \\ 1, & \text{if } n \equiv 1 \pmod{2}. \end{cases}$
4. $\gcd(F_{4n+2} - 4, F_{4n+3} - 4) = \begin{cases} 17, & \text{if } n \equiv 2 \pmod{9}; \\ 1, & \text{otherwise.} \end{cases}$

Theorem 3.3. *Let n be any integer. Then we have*

1. $\gcd(L_{4n-1} + 4, L_{4n} + 4) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod{2}; \\ 1, & \text{if } n \equiv 1 \pmod{2}. \end{cases}$
2. $\gcd(L_{4n} + 4, L_{4n+1} + 4) = \begin{cases} 11, & \text{if } n \equiv 4 \pmod{5}; \\ 1, & \text{otherwise.} \end{cases}$
3. $\gcd(L_{4n+1} + 4, L_{4n+2} + 4) = \begin{cases} 7, & \text{if } n \equiv 3 \pmod{4}; \\ 1, & \text{otherwise.} \end{cases}$

$$4. \gcd(L_{4n+2} + 4, L_{4n+3} + 4) = \begin{cases} 11, & \text{if } n \equiv 1 \pmod{5}; \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 3.4. *Let n be any integer. Then we have*

$$1. \gcd(L_{4n-1} - 4, L_{4n} - 4) = \begin{cases} 3, & \text{if } n \equiv 1 \pmod{2}; \\ 1, & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

$$2. \gcd(L_{4n} - 4, L_{4n+1} - 4) = 1.$$

$$3. \gcd(L_{4n+1} - 4, L_{4n+2} - 4) = \begin{cases} 7, & \text{if } n \equiv 1 \pmod{4}; \\ 1, & \text{otherwise.} \end{cases}$$

$$4. \gcd(L_{4n+2} - 4, L_{4n+3} - 4) = 1.$$

4. Proofs of main results. In this section, we give proof for the third item of Theorem 3.1 and first item of Theorem 3.2 as a showcase. The others could be proven similarly by Lemmas 2.1, 2.2 and 2.3.

Proof of Theorem 3.1. 3. In Lemma 2.3, let $m = 4n + 1$, $k = n$, $a = 4$. Then

$$(4.1) \quad \begin{aligned} \gcd(F_{4n+1} + 4, F_{4n+2} + 4) &= \gcd(F_{2n+1} + 4F_{2n-1}, -3F_{2n}) \\ &= \gcd(F_{2n} + 5F_{2n-1}, 3F_{2n}). \end{aligned}$$

By Lemma 2.1, we get

$$(4.2) \quad \gcd(F_{4n+1} + 4, F_{4n+2} + 4) = \gcd(F_{2n} + 5F_{2n-1}, 3 \gcd(F_{2n} + 5F_{2n-1}, F_{2n})).$$

Using Lemma 2.1 and Lemma 2.2 gives

$$(4.3) \quad \begin{aligned} \gcd(F_{2n} + 5F_{2n-1}, F_{2n}) &= \gcd(F_{2n}, 5F_{2n-1}) = \gcd(F_{2n}, 5) \\ &= \gcd(F_{2n}, F_5) = F_{\gcd(2n, 5)} \end{aligned}$$

and since

$$(4.4) \quad \gcd(2n, 5) = \begin{cases} 5, & \text{if } n \equiv 0 \pmod{5}; \\ 1, & \text{otherwise} \end{cases}$$

we get

$$(4.5) \quad F_{\gcd(2n, 5)} = \begin{cases} F_5 = 5, & \text{if } n \equiv 0 \pmod{5}; \\ F_1 = 1, & \text{otherwise.} \end{cases}$$

Then

$$(4.6) \quad \gcd(F_{4n+1} + 4, F_{4n+2} + 4) = \begin{cases} \gcd(F_{2n} + 5F_{2n-1}, 15), & \text{if } n \equiv 0 \pmod{5}; \\ \gcd(F_{2n} + 5F_{2n-1}, 3), & \text{otherwise.} \end{cases}$$

Thus the claim follows from Lemma 2.4. $\square$

Proof of Theorem 3.2. 1. In Lemma 2.3, let $m = 4n - 1$, $k = n$, $a = -4$. Then

$$(4.7) \quad \begin{aligned} \gcd(F_{4n-1} - 4, F_{4n} - 4) &= \gcd(-3F_{2n-1}, F_{2n-2} + 4F_{2n}) \\ &= \gcd(3F_{2n-1}, 5F_{2n} - F_{2n-1}). \end{aligned}$$

By Lemma 2.1, we get

$$(4.8) \quad \begin{aligned} \gcd(F_{4n-1} - 4, F_{4n} - 4) &= \gcd(15F_{2n}, 5F_{2n} - F_{2n-1}) \\ &= \gcd(15 \gcd(F_{2n}, 5F_{2n} - F_{2n-1}), 5F_{2n} - F_{2n-1}). \end{aligned}$$

With Lemma 2.1 and Lemma 2.2, $\gcd(F_{2n}, 5F_{2n} - F_{2n-1}) = 1$. Hence

$$(4.9) \quad \gcd(F_{4n-1} - 4, F_{4n} - 4) = \gcd(15, 5F_{2n} - F_{2n-1}).$$

Therefore, the claim follows with Lemma 2.5. $\square$

5. Conclusions Recently, there has been an increasing interest in the greatest common divisors of shifted Fibonacci (Lucas) numbers formed by adding an integer a to Fibonacci (Lucas) numbers, and in the studies conducted, the greatest common divisors have been investigated for $a = \pm 1, \pm 2, \pm 3$. In this paper, the greatest common divisor values were examined for consecutive Fibonacci and Lucas numbers shifted by $a = \pm 4$. We noticed that $\gcd(F_n \pm 4, F_{n+1} \pm 4)$ and $\gcd(L_n \pm 4, L_{n+1} \pm 4)$ are bounded. Studies that can be done on different number sequences in this direction will make a significant contribution to the literature. The focus of earlier studies has primarily been on the Fibonacci and Lucas numbers. Future researchers are recommended to examine other integer sequences, such as Pell and Pell–Lucas numbers to find out about their unboundedness or boundedness.

REFERENCES

- [1] HORADAM A. F. (1961) A generalized Fibonacci sequence, *Am. Math. Mon.*, **68**, 455–459.
- [2] HORADAM A. F. (1963) Complex Fibonacci numbers and Fibonacci quaternions, *Am. Math. Mon.*, **70**, 289–291.
- [3] TASCI D. (2021) Bicomplex balancing and bicomplex Lucas-balancing quaternions, *J. Sci. Arts*, **1**, 57–66.
- [4] ARSLAN H. (2021) Gaussian Pell and Gaussian Pell–Lucas quaternions, *Filomat*, **35**(5), 1609–1617.
- [5] TAŞDEMİR F., T. G. TOSKA (2020) Formulas for binomial double sums related to Lucas numbers, *Ars Combin.*, **152**, 235–246.
- [6] TAŞDEMİR F. (2021) Triple sums including Fibonacci numbers with three binomial coefficients, *NNTDM*, **27**(1), 188–197.

- [7] ADEGOKE K., R. FRONTCAK, T. GOY (2023) Binomial sum relations involving Fibonacci and Lucas numbers, *Appl. Math.*, **3**(4), 851–881.
- [8] TUĞLU N., S. KUŞ, C. KIZILATES (2024) A study of harmonic Fibonacci polynomials associated with Bernoulli-F and Euler–Fibonacci polynomials, *Indian J. Pure Appl. Math.*, **55**, 1129–1141.
- [9] KOSHY T. (2001) *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, Inc.
- [10] DUDLEY U., B. TUCKER (1971) Greatest common divisors in altered Fibonacci sequences, *Fibonacci Quart.*, **9**, 89–91.
- [11] CHEN K. W. (2011) Greatest common divisors in shifted Fibonacci sequences, *J. Integer Seq.*, **14**, Article 11.4.7.
- [12] RAHN A., M. KREH (2018) Greatest common divisors of shifted Fibonacci sequences revisited, *J. Integer Seq.*, **21**, Article 18.6.7.
- [13] KOKEN F. (2020) The gcd sequences of the altered Lucas sequences, *Ann. Math. Sil.*, **34**, 222–240.
- [14] SPILKER J. (2016) The GCD of the Shifted Fibonacci Sequence. In: *From Arithmetic to Zeta-Functions* (eds J. Sander, J. Steuding, R. Steuding), Springer, 473–483.
- [15] CHEN K. W., Y. R. PAN (2020) Greatest common divisors of shifted Horadam sequences, *J. Integer Seq.*, **23**, Article 20.5.8.

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