

DIAMETER ESTIMATES FOR COMPLETE  
 $(\tau, \rho)$ -QUASI RICCI HARMONIC SOLITONS

Yasemin Soylu<sup>✉</sup>, Davut Karaköse

Received on March 13, 2025

Presented by S. Ivanov, Member of BAS, on May 27, 2025

Abstract

In this paper, we prove that a complete and connected  $(\tau, \rho)$ -quasi Ricci harmonic soliton under certain conditions becomes compact and includes a diameter estimate. This result generalizes the theorem of Galloway. Also, we obtained that the  $(\tau, \rho)$ -quasi Ricci harmonic soliton has a finite fundamental group.

**Key words:** Bochner formula, diameter estimate, fundamental group, Ricci harmonic soliton

**2020 Mathematics Subject Classification:** Primary 53C20, secondary 53C21

**1. Introduction.** Let  $(M, g)$  and  $(N, h)$  be closed Riemannian manifolds of dimensions  $n$  and  $k$ , respectively. Let  $\phi(t) : M \rightarrow N$  be a smooth map from  $M$  to  $N$ . MÜLLER [1] introduced the following Ricci-harmonic flow as

$$(1.1) \quad \begin{cases} \frac{\partial g(t)}{\partial t} = -2\text{Ric}_{g(t)} + 2\alpha(t)\nabla\phi(t) \otimes \nabla\phi(t), & g(0) = g_0, \\ \frac{\partial \phi(t)}{\partial t} = \tau_{g(t)}\phi(t), & \phi(0) = \phi_0, \end{cases}$$

where  $g(t)$  is a time-dependent metric on  $M$ ,  $\nabla\phi \otimes \nabla\phi := \phi^*h$  is the pull-back of the metric  $h$  via  $\phi$ ,  $\tau_g\phi = \text{trace } \nabla d\phi$  is the tension field of  $\phi$  with respect to  $g$  and  $\alpha$  is a positive constant (possibly time-dependent).

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<https://doi.org/10.7546/CRABS.2025.07.02>

On a Riemannian manifold  $(M, g)$ , Ricci-harmonic solitons are self-similar solutions of the Ricci-harmonic flow. Let  $f : M \rightarrow \mathbb{R}$  be a smooth function, and let  $\phi : M \rightarrow N$  be a smooth map (not necessarily a harmonic map). We call  $(M, g, f, \phi)$  a *gradient Ricci-harmonic soliton* if

$$(1.2) \quad \begin{cases} \text{Ric}_f - \alpha \nabla \phi \otimes \nabla \phi = \mu g, \\ \tau_g \phi = g(\nabla \phi, \nabla f), \end{cases}$$

where  $\mu$  is a soliton constant and  $\text{Ric}_f$  denotes the Bakry–Émery curvature, defined as

$$\text{Ric}_f = \text{Ric} + \nabla^2 f.$$

We say that the soliton is shrinking, steady or expanding, if  $\mu > 0$ ,  $\mu = 0$  or  $\mu < 0$ , respectively.

When  $f$  is constant in (1.2), the gradient Ricci-harmonic soliton is reduced to the *harmonic-Einstein soliton* [2], i.e.,

$$(1.3) \quad \begin{cases} \text{Ric} - \alpha \nabla \phi \otimes \nabla \phi = \mu g, \\ \tau_g \phi = 0. \end{cases}$$

Also, if  $(N, h) = (\mathbb{R}, dr^2)$  and  $\phi : (M, g) \rightarrow (\mathbb{R}, dr^2)$  is a constant function in (1.2), then the gradient Ricci-harmonic soliton is reduced to the gradient Ricci soliton, see [3].

**Definition 1.1.** Let  $\phi : M \rightarrow N$  be a smooth map, and  $f : M \rightarrow \mathbb{R}$  be a potential function. A  $\tau$ -quasi Ricci-harmonic soliton structure on  $M$  satisfies the following system

$$(1.4) \quad \begin{cases} \text{Ric}_{f,\tau} - \alpha \nabla \phi \otimes \nabla \phi = \mu g, \\ \tau_g \phi = g(\nabla \phi, \nabla f) \end{cases}$$

for  $\alpha \geq 0$ ,  $\tau > 0$  and some constant  $\mu \in \mathbb{R}$ .

Here, the  $\tau$ -Bakry–Émery Ricci tensor,  $\text{Ric}_{f,\tau}$ , given by

$$\text{Ric}_{f,\tau} = \text{Ric} + \nabla^2 f - \frac{1}{\tau} df \otimes df.$$

Note that, if  $(N, h) = (\mathbb{R}, dr^2)$  and  $\phi : (M, g) \rightarrow (\mathbb{R}, dr^2)$  is a constant function, then the  $\tau$ -quasi Ricci-harmonic soliton is reduced to the  $\tau$ -quasi Einstein soliton, see [4].

In this paper, we denote generalized Ricci curvature by

$$\text{Ric}_\phi := \text{Ric} - \alpha \nabla \phi \otimes \nabla \phi,$$

and the metric trace of  $\text{Ric}_\phi$  by

$$R_\phi := R - \alpha |\nabla \phi|^2.$$

In the case the constant  $\mu$  right-hand side of equation (1.4) is replaced with  $\rho R_\phi + \mu$ , such as  $\rho \in \mathbb{R}$ , then the Definition 1.1 is replaced with

**Definition 1.2.** Let  $\phi : M \rightarrow N$  be a smooth map, and  $f : M \rightarrow \mathbb{R}$  be a potential function. A  $(\tau, \rho)$ -quasi Ricci-harmonic soliton structure on  $M$  satisfies the following system

$$(1.5) \quad \begin{cases} \text{Ric}_{f,\tau} - \alpha \nabla \phi \otimes \nabla \phi = (\rho R_\phi + \mu)g, \\ \tau_g \phi = g(\nabla \phi, \nabla f) \end{cases}$$

for  $\alpha \geq 0$ ,  $\tau > 0$  and some constant  $\mu, \rho \in \mathbb{R}$ .

Note that, if  $(N, h) = (\mathbb{R}, dr^2)$  and  $\phi : (M, g) \rightarrow (\mathbb{R}, dr^2)$  is a constant function, then the  $(\tau, \rho)$ -quasi Ricci-harmonic soliton is reduced to the  $(\tau, \rho)$ -quasi Einstein soliton, see [5, 6].

In recent years, Ricci solitons and their generalizations have been studied by several authors within the framework of compactness theorems and structural equations. In [7], authors proved a compactness theorem for a complete shrinking Ricci–Bourguignon harmonic soliton satisfying  $\text{Ric} + \mathcal{L}_X g - \alpha \nabla \phi \otimes \nabla \phi - \rho R \geq \lambda g$ ,  $\tau_g \phi = \mathcal{L}_X \phi$  with smooth vector field  $X$  having bounded norm. Some basic structural equations for almost Ricci-harmonic solitons, and almost Ricci–Bourguignon solitons were proved in [8] and [9]. SHAIKH et al. [10, Theorem 1.1] derived a diameter estimation for  $(\tau, \rho)$ -quasi Einstein manifolds. In [11], TADANO obtained some compactness theorems of Galloway-type for complete Riemannian manifolds in the context of the Bakry–Émery and modified Ricci curvatures. On the other hand, ZENG [12] obtained integral formulations and rigidity results for  $\tau$ -quasi Ricci-harmonic metrics, including explicit characterizations of generalized scalar curvature under constant conditions. Similar results were derived for the generalized  $\tau$ -quasi Ricci-harmonic metrics [13]. CAO [14] expanded the analysis of Ricci–Bourguignon harmonic solitons using  $V$ -harmonic map techniques.

**2. An extension of Galloway’s theorem for  $(\tau, \rho)$ -quasi Ricci-harmonic solitons.** In this section, we will use the notation

$$\text{Sc}_{f,\tau} := \text{Ric}_{f,\tau} - \alpha \nabla \phi \otimes \nabla \phi - (\rho R_\phi + \mu)g$$

to derive the compactness theorem for  $(\tau, \rho)$ -quasi Ricci-harmonic solitons.

Consider now the  $f$ -Laplacian operator  $\Delta_f$  on the complete weighted manifold  $(M, g, e^{-f} d\text{vol})$  such that

$$\Delta_f u = e^f \text{div} \left( e^{-f} \nabla u \right) = \Delta u - g(\nabla u, \nabla f),$$

where  $u$  is a smooth function on  $M$ . It is well known that the  $\text{Ric}_{f,\tau}$  connects with  $f$ -Laplacian by the following weighted Bochner formula [15]

$$\begin{aligned} \frac{1}{2}\Delta_f|\nabla u|^2 &= |\text{Hess}(u)|^2 + g(\nabla u, \nabla \Delta_f u) + \text{Ric}_f(\nabla u, \nabla u) \\ (2.1) \quad &= |\text{Hess}(u)|^2 + g(\nabla u, \nabla \Delta_f u) + \text{Ric}_{f,\tau}(\nabla u, \nabla u) + \frac{1}{\tau}|g(\nabla f, \nabla u)|^2. \end{aligned}$$

From equation (2.1), we have the following generalized Bochner formula for  $(\tau, \rho)$ -quasi Ricci-harmonic solitons:

$$\begin{aligned} \frac{1}{2}\Delta_f|\nabla u|^2 &= |\text{Hess}(u)|^2 + g(\nabla u, \nabla \Delta_f u) + \text{Sc}_{f,\tau}(\nabla u, \nabla u) + \frac{1}{\tau}|g(\nabla f, \nabla u)|^2 \\ (2.2) \quad &+ \alpha|g(\nabla \phi, \nabla u)|^2 + (\rho R_\phi + \mu)g(\nabla u, \nabla u). \end{aligned}$$

Using this formula, one can derive a generalization of a classical lemma that provides an estimate for the integral of the Ricci curvature along minimizing geodesics. The proof follows a structure inspired by earlier works [16].

**Lemma 2.1.** *Let  $(M, g, e^{-f} \text{dvol})$  be a complete weighted manifold, and consider the tensor  $\text{Sc}_{f,\tau}$ , such that  $(\rho R_\phi + \mu) \geq (\tau + n - 1)\sigma > 0$ . Let  $r$  be the distance function with respect to a fixed point  $p \in M$ , i.e.  $r(x) = \text{dist}(x, p)$ . Suppose that  $\beta : [0, r(q)] \rightarrow M$  is a minimal geodesic joining from  $p$  to  $q$ , parametrized by arc length. If  $\varphi \in \text{Lip}_{\text{loc}}(\mathbb{R})$  satisfies the boundary conditions  $\varphi(0) = \varphi(r(q)) = 0$ , then the inequality*

$$\begin{aligned} 0 \geq & -(\tau + n - 1) \int_0^{r(q)} (\varphi')^2 ds + \int_0^{r(q)} \varphi^2 \text{Sc}_{f,\tau}(\dot{\beta}, \dot{\beta}) ds \\ (2.3) \quad & + \sigma(\tau + n - 1) \int_0^{r(q)} \varphi^2 ds \end{aligned}$$

holds for all  $q \in M$ .

**Remark 2.2.** A function  $\varphi : [0, r(q)] \rightarrow \mathbb{R}$  is said to be uniformly Lipschitz continuous on  $[0, r(q)]$  (or *Lipschitz*, for short) if there exists a constant  $C$  such that

$$|\varphi(x) - \varphi(y)| \leq C|x - y| \quad \text{for all } x, y \in [0, r(q)].$$

The Lipschitz constant of  $\varphi$  is the infimum of constants  $C$  with this property. The space of Lipschitz functions on  $[0, r(q)]$  is denoted by  $\text{Lip}[0, r(q)]$ . The space of locally Lipschitz functions on  $\mathbb{R}$  is defined as

$$\text{Lip}_{\text{loc}}(\mathbb{R}) = \{\varphi : \mathbb{R} \rightarrow \mathbb{R} : \varphi \in \text{Lip}[0, r(q)] \text{ for all } 0 < r(q)\}.$$

Such functions need not be differentiable everywhere, but they still possess enough regularity for analytic techniques such as integration by parts. Therefore, they are suitable for the variational and geometric arguments employed in this paper.

**Proof.** Let us fix a point  $q \notin \text{cut}(p)$ . Direct calculations show that

$$(2.4) \quad \frac{(\Delta r)^2}{n-1} \leq |\text{Hess}(r)|^2,$$

$$(2.5) \quad \frac{(\Delta_f r)^2}{\tau+n-1} \leq \frac{(\Delta r)^2}{n-1} + \frac{|g(\nabla f, \nabla r)|^2}{\tau}.$$

Using inequalities (2.4), (2.5) and applying Bochner formula (2.2) to the distance function  $r(x)$ , we find that

$$(2.6) \quad \begin{aligned} 0 &\geq \frac{(\Delta_f r)^2}{\tau+n-1} + g(\nabla r, \nabla \Delta_f r) + \text{Sc}_{f,\tau}(\nabla r, \nabla r) + \alpha |g(\nabla \phi, \nabla r)|^2 \\ &\quad + (\rho R_\phi + \mu) g(\nabla r, \nabla r). \end{aligned}$$

Along a minimal geodesic  $\beta$  given in the lemma, we have

$$(2.7) \quad 0 \geq \frac{(\Delta_f r \circ \beta)^2}{\tau+n-1} + \frac{d}{ds}(\Delta_f r \circ \beta) + \text{Sc}_{f,\tau}(\dot{\beta}, \dot{\beta}) + \alpha |g(\nabla \phi, \dot{\beta})|^2 + (\rho R_\phi + \mu).$$

Multiplying (2.7) by  $\varphi^2$  and then integrating on  $[0, t]$ , we obtain

$$(2.8) \quad \begin{aligned} 0 &\geq \int_0^t \varphi^2 \frac{(\Delta_f r \circ \beta)^2}{\tau+n-1} ds + \int_0^t \varphi^2 \frac{d}{ds}(\Delta_f r \circ \beta) ds + \int_0^t \varphi^2 \text{Sc}_{f,\tau}(\dot{\beta}, \dot{\beta}) ds \\ &\quad + \int_0^t \varphi^2 (\rho R_\phi + \mu) ds, \end{aligned}$$

where  $\varphi \in \text{Lip}_{\text{loc}}(\mathbb{R})$ ,  $\varphi \geq 0$  and  $\varphi(0) = 0$ .

Integrating by parts and using  $(\Delta_f r \circ \beta_q) \varphi^2 \rightarrow 0$  as  $r \rightarrow 0$ , we obtain

$$(2.9) \quad \begin{aligned} 0 &\geq \int_0^t \frac{\varphi^2 (\Delta_f r \circ \beta)^2}{\tau+n-1} ds + \varphi^2(t) (\Delta_f r \circ \beta)(t) - 2 \int_0^t \varphi \varphi' (\Delta_f r \circ \beta) ds \\ &\quad + \int_0^t \varphi^2 \text{Sc}_{f,\tau}(\dot{\beta}, \dot{\beta}) ds + \sigma(\tau+n-1) \int_0^t \varphi^2 ds \end{aligned}$$

under the assumption of  $\rho R_\phi + \mu \geq (\tau+n-1)\sigma > 0$ .

Since  $X = (\tau+n-1)\varphi'^2 \geq 0$  and  $Y = \frac{\varphi^2}{\tau+n-1}(\Delta_f r \circ \beta)^2 \geq 0$ , we have the inequality  $\sqrt{XY} \leq \frac{1}{2}(X+Y)$ , i.e.,

$$(2.10) \quad \varphi \varphi' (\Delta_f r \circ \beta) \leq \frac{1}{2} \left( \frac{\varphi^2 (\Delta_f r \circ \beta)^2}{\tau+n-1} + (\tau+n-1)(\varphi')^2 \right).$$

Therefore, we have

$$(2.11) \quad \begin{aligned} 0 &\geq \varphi^2(t)(\Delta_{f^r} \circ \beta)(t) - \int_0^t (\tau + n - 1)(\varphi')^2 ds + \int_0^t \varphi^2 \operatorname{Sc}_{f,\tau}(\dot{\beta}, \dot{\beta}) ds \\ &\quad + \sigma(\tau + n - 1) \int_0^t \varphi^2 ds. \end{aligned}$$

In addition, if  $\varphi$  satisfies  $\varphi^2(r(q)) = 0$ , then the inequality (2.11) becomes

$$(2.12) \quad \begin{aligned} 0 &\geq -(\tau + n - 1) \int_0^{r(q)} (\varphi')^2 ds + \int_0^{r(q)} \varphi^2 \operatorname{Sc}_{f,\tau}(\dot{\beta}, \dot{\beta}) ds \\ &\quad + \sigma(\tau + n - 1) \int_0^{r(q)} \varphi^2 ds. \end{aligned}$$

This completes the proof for  $q \notin \operatorname{cut}(p)$ . To extend inequality (2.12) to any  $q \in M$ , we can use the Calabi trick. Assume  $q \in \operatorname{cut}(p)$ . Translating the origin  $p$  to  $p_\varepsilon = \beta_q(\varepsilon)$  so that  $q \notin \operatorname{cut}(p_\varepsilon)$ , and applying the triangle inequality, we conclude that inequality (2.12) remains valid in the limit as  $\varepsilon \rightarrow 0$ .  $\square$

We now give a generalization of GALLOWAY's compactness theorem [17] by utilizing the above lemma. The following is our first main result.

**Theorem 2.3.** *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. If there exist some constants  $c$  and  $K \geq 0$ , and a minimal geodesic  $\beta$  parametrized by arc length joining every pair of points in  $M$  such that*

$$(2.13) \quad \operatorname{Sc}_{f,\tau}(\dot{\beta}, \dot{\beta}) \geq (\tau + n - 1) \left( c^2 + \frac{d}{dt}(F \circ \beta) \right)$$

*holds for some smooth function  $F$  with  $|F| \leq K$  and  $(\rho R_\phi + \mu) \geq (\tau + n - 1)\sigma > 0$ , then  $M$  is compact and*

$$(2.14) \quad \operatorname{diam}(M) \leq \frac{1}{c^2 + \sigma} \left( 2K + \sqrt{4K^2 + (c^2 + \sigma)\pi^2} \right).$$

**Proof.** Let  $p, q$  be distinct points in  $M$  and  $\beta$  be a minimal geodesic segment from  $p$  to  $q$  of length  $\ell$ . By setting the function  $\varphi(t)$  to be  $\varphi(t) := \sin\left(\frac{\pi}{\ell}t\right) \in \operatorname{Lip}_{\operatorname{loc}}(\mathbb{R})$  as it is infinitely differentiable and hence Lipschitz on every compact interval such that  $\varphi(0) = \varphi(\ell) = 0$ . Using the assumption (2.13) and the inequality (2.3) in Lemma 2.1, we get

$$(2.15) \quad \begin{aligned} 0 &\geq -(\tau + n - 1) \int_0^\ell (\varphi')^2 dt + (\tau + n - 1)\sigma \int_0^\ell \varphi^2 dt + c^2(\tau + n - 1) \int_0^\ell \varphi^2 dt \\ &\quad + (\tau + n - 1) \int_0^\ell \varphi^2 \frac{d}{dt}(F \circ \beta) dt. \end{aligned}$$

Here, the assumption  $|F| \leq K$  implies

$$\begin{aligned}
 & (\tau + n - 1) \int_0^\ell \varphi^2 \frac{d}{dt} (F \circ \beta) dt \\
 &= (\tau + n - 1) \varphi^2 (F \circ \beta)(t) \Big|_0^\ell - (\tau + n - 1) \int_0^{\ell/2} \left( \frac{d}{dt} \varphi^2 \right) (F \circ \beta) dt \\
 &\quad - (\tau + n - 1) \int_{\ell/2}^\ell \left( \frac{d}{dt} \varphi^2 \right) (F \circ \beta) dt \\
 &\geq -\frac{(\tau + n - 1)K\pi}{\ell} \left[ \int_0^{\ell/2} \left| \sin \left( \frac{2\pi}{\ell} t \right) \right| dt + \int_{\ell/2}^\ell \left| \sin \left( \frac{2\pi}{\ell} t \right) \right| dt \right] \\
 (2.16) \quad &= -2K(\tau + n - 1).
 \end{aligned}$$

Inserting (2.16) into (2.15) and using

$$\begin{aligned}
 \int_0^\ell \varphi^2(t) dt &= \int_0^\ell \sin^2 \left( \frac{\pi}{\ell} t \right) dt = \frac{\ell}{2}, \\
 \int_0^\ell \varphi'^2(t) dt &= \frac{\pi^2}{\ell^2} \int_0^\ell \cos^2 \left( \frac{\pi}{\ell} t \right) dt = \frac{\pi^2}{2\ell},
 \end{aligned}$$

we obtain

$$(2.17) \quad 0 \geq -\frac{(\tau + n - 1)\pi^2}{2\ell} + \frac{(\tau + n - 1)(\sigma + c^2)\ell}{2} - 2K(\tau + n - 1).$$

Let us rewrite (2.17), we have

$$(2.18) \quad (c^2 + \sigma)\ell^2 - 4K\ell - \pi^2 \leq 0.$$

This completes the proof of the theorem. □

The second diameter estimate follows from WYLIE's theorem in [18].

**Theorem 2.4.** *Let  $(M, g)$  be a complete Riemannian manifold satisfying*

$$(2.19) \quad \begin{cases} \text{Ric}_{f,\tau} - \alpha \nabla \phi \otimes \nabla \phi \geq (\rho R_\phi + \mu)g, \\ \tau_g \phi = g(\nabla \phi, \nabla f) \end{cases}$$

*for  $\alpha \geq 0$ ,  $\tau > 0$  and some constant  $\mu \in \mathbb{R}$ . If  $\rho R_\phi + \mu \geq \sigma$  for a constant  $\sigma$  such that  $\sigma > 0$ , then for any  $p, q \in M$  we have*

$$(2.20) \quad d(p, q) \leq \max \left\{ 1, \frac{1}{\sigma} (2(n-1) + \mathcal{G}_p + \mathcal{G}_q + \|\nabla f\|_p + \|\nabla f\|_q) \right\}.$$

**Proof.** Let us suppose that  $d(p, q) > 1$  and  $\beta$  is the minimal geodesic from  $p$  to  $q$  parametrized by arc length. From inequality (2.19) and assumption  $\rho R_\phi + \mu \geq \sigma > 0$ , we get

$$\begin{aligned}
 & \int_0^r \text{Ric}(\dot{\beta}(s), \dot{\beta}(s)) ds \\
 & \geq - \int_0^r (\nabla^2 f)(\dot{\beta}(s), \dot{\beta}(s)) ds + \int_0^r \frac{1}{\tau} (df \otimes df)(\dot{\beta}(s), \dot{\beta}(s)) ds \\
 & + \alpha \int_0^r ((\nabla \phi \otimes \nabla \phi)(\dot{\beta}(s), \dot{\beta}(s)) ds + \int_0^r (\rho R_\phi + \mu) g(\dot{\beta}(s), \dot{\beta}(s)) ds \\
 & = - \int_0^r \frac{d}{ds} g(\nabla f, \dot{\beta}(s)) ds + \int_0^r (\rho R_\phi + \mu) g(\dot{\beta}(s), \dot{\beta}(s)) ds \\
 (2.21) \quad & \geq g_p(\nabla f, \dot{\beta}(0)) - g_q(\nabla f, \dot{\beta}(r)) + \sigma d(p, q).
 \end{aligned}$$

On the other hand, since  $|\dot{\beta}| = 1$ , we also have  $-g_p(\nabla f, \dot{\beta}(0)) \leq \|\nabla f\|_p$  and  $g_q(\nabla f, \dot{\beta}(r)) \leq \|\nabla f\|_q$  by using the Cauchy–Schwarz inequality. Therefore, the above inequality becomes

$$\int_0^r \text{Ric}(\dot{\beta}(s), \dot{\beta}(s)) ds \geq \sigma d(p, q) - \|\nabla f\|_p - \|\nabla f\|_q.$$

Recall from [18, Lemma 2.2] that

$$\int_0^r \text{Ric}(\dot{\beta}(s), \dot{\beta}(s)) ds \leq 2(n-1) + \mathcal{G}_p + \mathcal{G}_q,$$

where  $\mathcal{G}_p = \max \{0, \sup \{\text{Ric}_y(v, v) : y \in B(p, 1), \|v\| = 1\}\}$ . Thus

$$2(n-1) + \mathcal{G}_p + \mathcal{G}_q \geq \sigma d(p, q) - \|\nabla f\|_p - \|\nabla f\|_q.$$

We solve for  $d(p, q)$  to get

$$1 < d(p, q) \leq \frac{1}{\sigma} (2(n-1) + \mathcal{G}_p + \mathcal{G}_q + \|\nabla f\|_p + \|\nabla f\|_q),$$

which completes the proof of the theorem.  $\square$

The following result for the Ricci solitons and Ricci–Bourguignon harmonic solitons was proved in [18] and [7], respectively, which is a natural corollary of Theorem 2.4.

**Corollary 2.5.** *Let  $((M, g), (N, h), \phi, \nabla f, \rho)$  be a complete shrinking  $(\tau, \rho)$ -quasi Ricci-harmonic soliton. Let  $\rho R_\phi + \mu \geq \sigma$  for a constant  $\sigma$  such that  $\sigma > 0$ . Then  $M$  has finite fundamental group.*

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*Giresun University, Department of Mathematics, 28100, Giresun, Turkey*  
*e-mails: yasemin.soylu@giresun.edu.tr, o232110005@giresun.edu.tr*