

CHARACTERIZATION OF EXCHANGEABLE HOEFFDING-DECOMPOSABLE SEQUENCES

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Abstract

Hoeffding decomposition is a classical method for studying symmetric statistics based on i.i.d. observations. The notion was extended by G. Peccati [*Ann. Probab.* **32** (2004) 1796–1829] under exchangeability, but apart from a few notable examples, the question of which exchangeable processes are Hoeffding-decomposable remains open. In this note, we announce a complete characterization of the class of exchangeable Hoeffding-decomposable processes in terms of their predictive distributions. As a result, we show that they are connected to exchangeable measure-valued Pólya urn sequences, from which we derive the general form of their prior distributions.

Key words: Hoeffding decomposition, exchangeability, Pólya urns, predictive distributions

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1. Introduction and preliminaries. More than 20 years ago, G. PECCATI [1,2] introduced the concept of an exchangeable Hoeffding-decomposable sequence and later used it to establish several new spectral properties of the Dirichlet process, including a chaotic representation of functionals of the Dirichlet process [3]. Despite all subsequent developments, see, e.g., [4,5], the question of which exchangeable processes are Hoeffding-decomposable remains open. This note provides a complete and concise answer, and serves as an announcement of the main results in [6].

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Suppose that $(\mathbb{X}, \mathcal{X})$ is a standard Borel space. Let $(X_n)_{n \geq 1}$ be an (infinite) exchangeable sequence of $\mathbb{X}$ -valued random variables. By de Finetti's representation theorem for exchangeable sequences [7, Theorem 3.1], there exists a random probability measure $\tilde{P}$ on $\mathbb{X}$, called the *directing random measure* of $(X_n)_{n \geq 1}$, such that, given $\tilde{P}$, the random variables $X_1, X_2, \dots$ are conditionally independent and identically distributed (i.i.d.) with marginal distribution $\tilde{P}$,

$$\begin{array}{ccc} X_1, X_2, \dots & | & \tilde{P} \stackrel{i.i.d.}{\sim} \tilde{P} \\ & & \tilde{P} \sim Q, \end{array}$$

where Q is the *prior* distribution of $\tilde{P}$.

For each $n \in \mathbb{N}$, we introduce the notation $\mathbf{x}_{1:n} = (x_1, \dots, x_n)$ and $\mathbb{P}_{\mathbf{x}_{1:n}}(\cdot) = \mathbb{P}(\mathbf{X}_{1:n} \in \cdot)$. Let us further denote by $L^2(\mathbb{P}_{\mathbf{x}_{1:n}})$ and $L_s^2(\mathbb{P}_{\mathbf{x}_{1:n}})$ the space of all $\mathbb{P}_{\mathbf{x}_{1:n}}$ -square-integrable functions and the subspace of all symmetric $\mathbb{P}_{\mathbf{x}_{1:n}}$ -square-integrable functions, respectively. Define $SU_0 := \mathbb{R}$ and, for $k = 1, \dots, n$,

$$SU_k(\mathbf{X}_{1:n}) := \left\{ T : T(\mathbf{x}_{1:n}) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \varphi(x_{j_1}, \dots, x_{j_k}), \varphi \in L_s^2(\mathbb{P}_{\mathbf{x}_{1:k}}) \right\}.$$

By Corollaries 4 and 5 in [2], each $SU_k(\mathbf{X}_{1:n})$ is an L_s^2 -closed vector space and, since $SU_k(\mathbf{X}_{1:n}) \subseteq SU_{k+1}(\mathbf{X}_{1:n})$, letting

$$SH_0 = SU_0 \quad \text{and} \quad SH_k(\mathbf{X}_{1:n}) = SU_k(\mathbf{X}_{1:n}) \cap SU_{k-1}(\mathbf{X}_{1:n})^\perp, \quad \text{for } k = 1, \dots, n,$$

we obtain the orthogonal decomposition $L_s^2(\mathbb{P}_{\mathbf{x}_{1:n}}) = \bigoplus_{k=0}^n SH_k(\mathbf{X}_{1:n})$. Therefore, every $T \in L_s^2(\mathbb{P}_{\mathbf{x}_{1:n}})$ admits an essentially unique representation

$$(1) \quad T = \mathbb{E}[T(\mathbf{X}_{1:n})] + \sum_{k=1}^n T_k,$$

where $T_k(\mathbf{x}_{1:n}) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \varphi_k(x_{j_1}, \dots, x_{j_k})$, for each $k = 1, \dots, n$ and some $\varphi_k \in L_s^2(\mathbb{P}_{\mathbf{x}_{1:k}})$, with $\mathbb{E}[T_k(\mathbf{X}_{1:n}) \cdot T_j(\mathbf{X}_{1:n})] = 0$, for $j \neq k$. If, in addition, the φ_k 's are *degenerate*, i.e. $\mathbb{E}[\varphi_k(\mathbf{X}_{1:k}) | \mathbf{X}_{1:k-1}] = 0$ a.s., we call (1) the *Hoeffding decomposition* of T . However, as [2, Example on p. 1807] shows, not every exchangeable process is Hoeffding-decomposable, which leads to the following definition.

Definition 1 (Definition 1 in [2]). We say that an exchangeable process $(X_n)_{n \geq 1}$ is Hoeffding-decomposable if, for every $n \in \mathbb{N}$ and $k = 1, \dots, n$, the following *double* implication holds:

$$T \in SH_k(\mathbf{X}_{1:n})$$

if and only if there exists $\varphi \in L_s^2(\mathbb{P}_{\mathbf{x}_{1:k}})$ such that

$$T(\mathbf{x}_{1:n}) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \varphi(x_{j_1}, \dots, x_{j_k}),$$

and

$$\mathbb{E}[\varphi(\mathbf{X}_{1:k}) | \mathbf{X}_{1:k-1}] = 0 \quad \text{a.s.},$$

which should be understood as an unconditional statement in the case $k = 1$.

2. Predictive characterization of Hoeffding decomposability. For any $f \in L^2(\mathbb{P}_{X_1})$, the conditional expectation $\mathbb{E}[f(X_{n+1})|\mathbf{X}_{1:n} = \mathbf{x}_{1:n}]$ is an element of $L^2_s(\mathbb{P}_{\mathbf{X}_{1:n}})$ under exchangeability, see, e.g., [8, Proposition 3.2], so it admits the representation (1). The next result provides a structural characterization of the predictive distributions of any exchangeable Hoeffding-decomposable sequence, showing that $\mathbb{E}[f(X_{n+1})|\mathbf{X}_{1:n} = \mathbf{x}_{1:n}]$ depends only on the first-order terms in its Hoeffding decomposition, and vice versa. The proof itself follows by demonstrating that the inner product of $f(X_{n+1})$ with any higher term is both zero and equal to the term's second moment. The assumption that $SH_k(\mathbf{X}_{1:n}) \neq \{0\}$, for all $k = 1, \dots, n$ and $n \in \mathbb{N}$, avoids trivialities and rules out *degenerate* processes, i.e. $X_n = X_1$ a.s., for all $n \in \mathbb{N}$, see also [2, p. 1803].

Proposition 2. Assume $SH_k(\mathbf{X}_{1:n}) \neq \{0\}$, for all $k = 1, \dots, n$ and $n \in \mathbb{N}$. An exchangeable process $(X_n)_{n \geq 1}$ is Hoeffding-decomposable if and only if, for every $n \in \mathbb{N}$ and $f \in L^2(\mathbb{P}_{X_1})$, there exists $\varphi_f^{(n)} \in L^2(\mathbb{P}_{X_1})$ such that

$$(2) \quad \mathbb{E}[f(X_{n+1})|\mathbf{X}_{1:n}] = \mathbb{E}[f(X_1)] + \sum_{i=1}^n \varphi_f^{(n)}(X_i) \quad a.s.$$

It follows from Proposition 2 that, for each $n \in \mathbb{N}$,

$$\Lambda_n f := \varphi_f^{(n)}, \quad f \in L^2(\mathbb{P}_{X_1}),$$

defines a linear operator on $L^2(\mathbb{P}_{X_1})$, in which case (2) becomes

$$(3) \quad \mathbb{E}[f(X_{n+1})|\mathbf{X}_{1:n}] = \mathbb{E}[f(X_1)] + \sum_{i=1}^n \Lambda_n f(X_i) \quad a.s.$$

Therefore, the problem of characterizing the class of exchangeable Hoeffding-decomposable sequences transforms into one of determining which exchangeable processes have predictive rules (3), which is the subject of the following theorem.

Theorem 3. Let $(X_n)_{n \geq 1}$ be an exchangeable sequence of $\mathbb{X}$ -valued random variables such that $X_1 \sim \nu$. The following are equivalent:

- 1) For each $n \in \mathbb{N}$, there is a linear operator Λ_n on $L^2(\nu)$ such that

$$\mathbb{E}[f(X_{n+1})|\mathbf{X}_{1:n}] = \mathbb{E}[f(X_1)] + \sum_{i=1}^n \Lambda_n(f)(X_i) \quad a.s., \text{ for every } f \in L^2(\nu).$$

- 2) There exist (i) a measurable function $\eta : \mathbb{X} \rightarrow \mathbb{R}$ satisfying $\eta(X_1) = \eta(X_2)$ a.s., (ii) a measurable function $g : \mathbb{R} \rightarrow [0, 1]$, and (iii) a probability kernel R on $\mathbb{X}$ such that

$$(4) \quad \begin{aligned} \mathbb{P}(X_{n+1} \in \cdot | \mathbf{X}_{1:n}) &= \frac{1 - g(\eta(X_1))}{1 + (n-1)g(\eta(X_1))} \nu(\cdot | \sigma(\eta))(X_1) \\ &+ \frac{g(\eta(X_1))}{1 + (n-1)g(\eta(X_1))} \sum_{i=1}^n R_{X_i}(\cdot) \quad a.s., \end{aligned}$$

where $\nu(\cdot | \sigma(\eta))$ denotes a regular version of the conditional distribution for ν given $\sigma(\eta)$.

The implication from 2) to 1) in Theorem 3 follows immediately. It is the implication from 1) to 2) that requires much more effort, necessitating a careful and systematic study of the operators Λ_n with tools from functional analysis and measure theory, making use of the structure and properties of the exchangeable process $(X_n)_{n \geq 1}$. In particular, through a long series of calculations and using [8, Proposition 3.2], we show that $\Lambda_n = (I + (n-1)\Lambda_1)^{-1}\Lambda_1$, for all $n \in \mathbb{N}$, and $\Lambda_1 h = c \cdot h$ a.e. $[\nu]$, for all h in a certain subclass $\mathcal{M} \subseteq L^2(\nu)$ and some $c \in L^2(\nu)$ such that $c \in [0, 1]$ a.e. $[\nu]$ and $c(X_n) = c(X_1)$ a.s., for each $n = 2, 3, \dots$. The proof is then completed by constructing η , g , and R with the necessary properties from c and the image of Λ_1 on $\mathcal{M}^\perp$.

3. Measure-valued Pólya urn sequences and Hoeffding decomposability. A sequence $(X_n)_{n \geq 1}$ of $\mathbb{X}$ -valued random variables is said to be a *measure-valued Pólya sequence* [9] with parameters θ , ν and R , denoted MVPS(θ, ν, R), if $X_1 \sim \nu$ and, for each $n \in \mathbb{N}$,

$$(5) \quad \mathbb{P}(X_{n+1} \in \cdot | \mathbf{X}_{1:n}) = \frac{\theta \nu(\cdot) + \sum_{i=1}^n R_{X_i}(\cdot)}{\theta + \sum_{i=1}^n R_{X_i}(\mathbb{X})},$$

where $\theta > 0$, ν is a probability measure on $\mathbb{X}$, and R is a finite transition kernel on $\mathbb{X}$. If $(X_n)_{n \geq 1}$ is exchangeable, by Remark 3.3. and Corollary 3.4 in [9], we can assume without loss of generality that R is a probability kernel on $\mathbb{X}$, in which case (5) becomes

$$(6) \quad \mathbb{P}(X_{n+1} \in \cdot | \mathbf{X}_{1:n}) = \frac{\theta}{\theta + n} \nu(\cdot) + \frac{1}{\theta + n} \sum_{i=1}^n R_{X_i}(\cdot);$$

thus, we can easily see that every exchangeable MVPS satisfies (3). Moreover, by Theorem 3.9 in [9], $(X_n)_{n \geq 1}$ has directing random measure

$$(7) \quad \tilde{P}(\cdot) \stackrel{w}{=} \sum_{j \geq 1} V_j R_{U_j}(\cdot),$$

where $V_1 = W_1$ and $V_j = W_j \prod_{i=1}^{j-1} (1 - W_i)$, for $j \geq 2$, with $W_1, W_2, \dots \stackrel{i.i.d.}{\sim}$ Beta(1, θ), and $U_1, U_2, \dots \stackrel{i.i.d.}{\sim} \nu$ are independent of $(V_j)_{j \geq 1}$. When $R_x = \delta_x$, it is well known that $\tilde{P}$ in (7) has a Dirichlet process prior [10, p. 112], in which case $(X_n)_{n \geq 1}$ is simply known as a *Pólya sequence*.

On the other hand, from the obvious parallels between (4) and (6), we obtain the following corollary of Theorem 3.

Corollary 4. *If $(X_n)_{n \geq 1}$ is as in Theorem 3, letting $\tilde{\eta} = \eta(X_1)$, we have*

$$X_1, X_2, \dots | \tilde{\eta} \sim \begin{cases} \text{exch. MVPS}(\frac{1 - g(\tilde{\eta})}{g(\tilde{\eta})}, \nu(\cdot | \eta = \tilde{\eta}), R) & \text{if } g(\tilde{\eta}) > 0, \\ \text{i.i.d. } \nu(\cdot | \eta = \tilde{\eta}) & \text{if } g(\tilde{\eta}) = 0, \end{cases}$$

$$\tilde{\eta} \sim \nu_\eta$$

where $\nu_\eta := \nu \circ \eta^{-1}$.

It follows from Corollary 4 that every exchangeable process $(X_n)_{n \geq 1}$ satisfying (3) must be one of the following w.r.t. some random parameter $\tilde{\eta}$ and function g ,

- (a) (unconditionally) degenerate, i.i.d., or an MVPS;
- (b) conditionally given $\tilde{\eta}$, degenerate or i.i.d. when $g(\tilde{\eta}) = 0$;
- (c) conditionally given $\tilde{\eta}$, degenerate, i.i.d., or an exchangeable MVPS when $g(\tilde{\eta}) > 0$.

In fact, $(X_n)_{n \geq 1}$ will be an MVPS if and only if $\sigma(\tilde{\eta}) = \{\emptyset, \Omega\}$ a.e. $[\nu]$, which is true whenever $\mathbb{P}(X_2 \in \cdot \mid X_1)$ has full ν -support. In the general case, the directing random measure of $(X_n)_{n \geq 1}$ has the form

$$\tilde{P}(\cdot) \stackrel{w}{=} \nu(\cdot \mid \eta = \tilde{\eta}) \cdot \mathbb{1}_{\{g(\tilde{\eta})=0\}} + \tilde{P}_{\tilde{\eta}}(\cdot) \cdot \mathbb{1}_{\{g(\tilde{\eta})>0\}},$$

where

$$\tilde{P}_m(\cdot) \stackrel{w}{=} \sum_{j \geq 1} V_{j,m} R_{U_{j,m}}(\cdot), \quad \text{for all } m \in \mathbb{R} \text{ such that } g(m) > 0,$$

with $(V_{j,m})_{j \geq 1}$ and $(U_{j,m})_{j \geq 1}$ as in (7) w.r.t. the parameters $(\frac{1-g(m)}{g(m)}, \nu(\cdot \mid \eta = m))$; thus, we are able to perform a full Bayesian analysis on any exchangeable process characterized by (3). However, as [6] shows, condition (3) is rather restrictive, since, for example, among the class of species sampling sequences [11], and in particular the subclass of exchangeable processes driven by a homogeneous normalized completely random measure [10], the only ones whose predictive distributions have the form (3) are i.i.d. and Pólya sequences. Notably, this shows that even the exchangeable process with a two-parameter Poisson–Dirichlet (Pitman–Yor) process prior is not Hoeffding-decomposable. As a result, the conclusions reached by [3] on the Dirichlet process using Hoeffding decomposability can be extended at most to the class of exchangeable MVPSs or to some of the mixture models described above.

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