

GENERALIZED ENTROPIES OBTAINED
FROM EXTENSIVITY PRINCIPLE

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Abstract

We present a derivation of generalized entropy measures based on the thermodynamic principle of extensivity, independent of heuristic or purely phenomenological arguments. Starting from the scaling behaviour of entropy with particle number in the thermodynamic limit, we derive known entropy forms including the Boltzmann–Gibbs and Tsallis entropies, as well as the entropy classes characterized by the Hanel–Thurner scaling exponents α and β . A novel two-parameter entropy is introduced, which captures systems exhibiting both finite-size effects and constrained phase space dynamics. This entropy reduces to known forms in limiting cases and provides a potential framework for describing complex systems with strong correlations, such as biological networks or constrained quantum systems. Our approach emphasizes the role of relative fluctuations and asymptotic scaling in determining the functional form of entropy, suggesting a unified thermodynamic origin for a spectrum of generalized entropies. Possible applications and implications for statistical physics and information theory are discussed.

Key words: generalized entropy, extensivity principles, thermodynamic limit, Khinchin axioms

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1. Introduction. The concept of entropy, a measure of disorder or uncertainty in a system, has been a cornerstone of thermodynamics and statistical mechanics since the works of Boltzmann, Gibbs, and Shannon. Traditionally, entropy has been understood as an additive quantity: the total entropy of a composite system equals the sum of entropies of its independent subsystems. This additivity, formalized in the fourth Khinchin axiom, holds for systems in which interactions and correlations decay rapidly with distance, allowing for an extensive thermodynamic description.

However, in the past few decades, a growing body of evidence has revealed the inadequacy of additive entropy measures for describing systems characterized by long-range interactions, fractal geometries, strong correlations, or memory effects. Such systems include gravitational systems, plasmas, turbulent flows, complex networks, and biological systems. To address these cases, non-additive and generalized entropy forms have been proposed, extending beyond the Boltzmann–Gibbs framework.

A prominent example is TSALLIS entropy [1], which introduces a parameter q quantifying the degree of non-extensivity. Tsallis entropy has found applications in areas ranging from astrophysics to econophysics and biological systems. Other generalized entropies have been proposed, including RÉNYI entropy [2], SHARMA–MITTAL entropy [3], KANIADAKIS entropy [4], and the HANEL–TURNER entropy class [5], characterized by a pair of scaling exponents α and β that generalize entropy composition rules and asymptotic behaviours.

These generalizations typically relax the additivity axiom but still aim to preserve core thermodynamic properties, such as concavity, Lesche stability, and extensivity under certain conditions. Extensivity, in particular the proportionality of entropy to system size in the thermodynamic limit – remains a critical requirement for linking microscopic statistical behaviours to macroscopic thermodynamic laws.

In this paper, we explore how generalized entropy measures can be derived from the principle of extensivity itself, without invoking additivity or other heuristic axioms. We show that different functional forms of entropy arise naturally from the asymptotic scaling of the phase space volume and the relative fluctuations in large systems. In particular, we recover known entropies, such as the Boltzmann–Gibbs and Tsallis forms, as well as the entropy class described by the Hanel–Turner exponents α and β [5]. We further derive a novel two-parameter entropy that may be suitable for describing highly constrained or strongly correlated systems.

2. Theory. Consider a physical system that can be described by a countable set of states Ω . The probability for the i -th state occurring is denoted by p_i with the normalization condition $\sum_{i \in \Omega} p_i = 1$. Then we can assume that the total entropy of the system is given as the sum of all individual entropies $g(p_i)$

associated with the event

$$(1) \quad S = \sum_{i \in \Omega} g(p_i).$$

To get equilibrium states of a physical system, one needs to extremize the entropy (1) under the normalization constraint, i.e. one needs to find the minimum of the function

$$(2) \quad \Gamma(p_1, \dots, p_W) = \sum_{i \in \Omega} g(p_i) + \lambda \left(\sum_{i \in \Omega} p_i - 1 \right),$$

where $W = |\Omega|$ is the size of the phase space and λ is a Lagrange multiplier. The minimum of (2) is just the equiprobability distribution, namely, $p_i = 1/W$ for all $i \in \Omega$. Substituting this extremal distribution into (1) one obtains the equilibrium state entropy

$$(3) \quad S_{\text{eq}} = \sum_{i \in \Omega} g(1/W) = W g(1/W).$$

All entropies must satisfy several axioms, called the Khinchin axioms. There are three axioms for entropies that must always be satisfied:

Axiom K1. *The function $g(p_i)$ is a continuous function and will only depend on the probability.*

Axiom K2. *The entropy S will have a maximum if it holds $p_i = 1/W$ for all $i \in \Omega$.*

Axiom K3. *It must hold $g(0) = 0$ because an impossible event will not give any information.*

In case of an additive entropy there is an additional fourth Khinchin axiom:

Axiom K4. *If A and B are two physical systems, then the entropy of the system obtained by combining A and B is the sum of entropy of A and entropy of B .*

The only entropy in the form (1) that satisfies all axioms K1–K4 is the Boltzmann–Gibbs entropy

$$(4) \quad g_{\text{BG}}(p_i) = -p_i \ln(p_i)$$

and is widely used in Statistical Physics. The entropy (4) is valid if the interactions and correlations between particles of the physical system are negligible. However, if the system becomes strongly correlated, one may need other entropic forms than (4) by relaxing the additivity condition K4.

In this paper we will address non-additive entropies but will employ the fundamental concept of Thermodynamics of extensivity. Extensivity of a physical system is defined as such if N is the number of particles in the system and we are

in thermodynamic limit, i.e. when N is extremely large, then S_{eq} scales proportional to the particle number. Formally, extensivity means

$$(5) \quad \lim_{N \rightarrow \infty} \frac{S_{\text{eq}}}{N} = s,$$

where s is a constant independent on N . We will study only systems that have a huge number of particles such that extensivity principle (5) can be used. Entropy classes that satisfy thermodynamic extensivity on top of K1–K3 will be studied by regarding the asymptotic growth of phase space volume $W(N)$ with particle number. It is expected that W is strictly monotonically increasing with particle number towards the thermodynamic limit. Thus, we have p_i as being very small values in thermodynamic equilibrium, since $W(N)$ has always a very high value. Also, S_{eq} has a very high value due to proportionality to W according to (3). This allows us to use L'Hospital's rule for limits in (5):

$$(6) \quad \lim_{N \rightarrow \infty} \frac{1}{N} W(N) g\left(\frac{1}{W(N)}\right) = \lim_{N \rightarrow \infty} \left(W'(N) g\left(\frac{1}{W(N)}\right) - \frac{W'(N)}{W(N)} g'\left(\frac{1}{W(N)}\right) \right).$$

Here, the primes denote the derivation of functions by their corresponding arguments. Assuming that $W(N)$ grows faster than linear with particle number, $W'(N)$ will again tend to infinity in thermodynamic limit. The factor $g\left(\frac{1}{W(N)}\right) - \frac{1}{W(N)} g'\left(\frac{1}{W(N)}\right)$ tends to zero which is obvious by K3 and because of $\frac{1}{W(N)}$ approaching zero. Hence, we employ L'Hospital's rule in (6) again to obtain

$$(7) \quad s = \lim_{N \rightarrow \infty} \frac{W'(N)}{\left(g\left(\frac{1}{W(N)}\right) - \frac{1}{W(N)} g'\left(\frac{1}{W(N)}\right) \right)^{-1}} \\ = \lim_{N \rightarrow \infty} \frac{W''(N)}{\frac{-W'(N)}{W^3(N)} g''\left(\frac{1}{W(N)}\right) \left(g\left(\frac{1}{W(N)}\right) - \frac{1}{W(N)} g'\left(\frac{1}{W(N)}\right) \right)^{-2}}.$$

Using the identity (6), we can eliminate the variable s

$$\lim_{N \rightarrow \infty} \frac{\frac{W''(N)}{W'(N)} \left(g\left(\frac{1}{W(N)}\right) - \frac{1}{W(N)} g'\left(\frac{1}{W(N)}\right) \right)}{\frac{-W'(N)}{W^3(N)} g''\left(\frac{1}{W(N)}\right)} = 1$$

which implies that

$$(8) \quad \lim_{N \rightarrow \infty} \frac{W''(N)}{W'(N)} \left(g\left(\frac{1}{W(N)}\right) - \frac{1}{W(N)} g'\left(\frac{1}{W(N)}\right) \right) \\ = \lim_{N \rightarrow \infty} \frac{-W'(N)}{W^3(N)} g''\left(\frac{1}{W(N)}\right).$$

Then we define a new quantity

$$(9) \quad \alpha = \lim_{N \rightarrow \infty} \left(\frac{W''(N)W(N)}{W'^2(N)} - 1 \right)$$

and rearrange (8) to obtain

$$(10) \quad \lim_{N \rightarrow \infty} \left((\alpha + 1) \left(g \left(\frac{1}{W(N)} \right) - \frac{1}{W(N)} g' \left(\frac{1}{W(N)} \right) \right) + \frac{1}{W^2(N)} g'' \left(\frac{1}{W(N)} \right) \right) = 0.$$

The parameter α is linked to fluctuations within the physical system. The expression $\Pi(N) = \ln(W(N))$ is a generating function for statistical moments when it has integrands proportional to $e^{\mu N}$ with a chemical potential μ dependent on integration variables, i.e. it generates important statistical quantities like means or variances when deriving it. For instance, $\Pi'(N) = \frac{W'(N)}{W(N)}$ is the mean value of the chemical potential and $\Pi''(N) = \frac{W''(N)}{W(N)} - \left(\frac{W'(N)}{W(N)} \right)^2$ is the variance. Thus, we can rewrite (9) as

$$(11) \quad \alpha = \frac{\Pi''(N)}{\Pi'^2(N)}$$

and see immediately that α is a measure of relative fluctuations. Many physical systems, particularly non-interacting ones, will have vanishing relative fluctuations in thermodynamic limit and for these α can be set to zero. This special case will simplify (10), after introducing the abbreviation $x = 1/W(N)$ to

$$(12) \quad \lim_{N \rightarrow \infty} (g(x) - xg'(x) + x^2g''(x)) = 0.$$

Note that (10) and (12) hold not only for infinitely many particles, but also when the particle number is finite, but extremely large, in other words, for real physical systems. Therefore, it must hold for any value of phase space volume assuming very large particle numbers. We have now a differential equation (12) for the function $g(x)$ that can be easily solved to give $g(x) = -x \ln x$ and hence the Boltzmann–Gibbs entropy (4). We have now derived the Boltzmann–Gibbs–Shannon entropy without using the additivity axiom K4. Instead, we have assumed extensivity and vanishing relative fluctuations. Another interesting case appears when we assume that α will tend to a constant value in thermodynamic limit and will not depend explicitly on size of the phase space in this regime. Since we will always work toward the thermodynamic limit we will drop the limit.

LLL Physical Justification of equation (10) for Finite N : The derivation of the differential equation (10) relies on the thermodynamic limit ($N \rightarrow \infty$). Applying this asymptotic form to finite N regimes by treating α as a variable function $\alpha(x)$, while keeping the differential equation intact is a strong assumption. This step

essentially constitutes a motivated ansatz or effective model for finite-size scaling, rather than an ab initio derivation from microscopic dynamics. Clarifying that $\alpha(x)$ acts as an effective parameter capturing finite-size corrections would make the logical flow more robust. This yields the differential equation

$$(1 + \alpha)(g(x) - xg'(x)) + x^2g''(x) = 0$$

which has the solution

$$(13) \quad g(x) = c_1x + c_2x^{\alpha+1}.$$

Using the initial conditions $g(0) = 0$, $g(1) = 0$ which come from axiom K3 and the fact that one certain event will not have any uncertainty, one obtains the Tsallis entropy

$$(14) \quad g_T(p_i) = p_i \frac{(1 - p_i^\alpha)}{\alpha}.$$

This is a common non-additive entropy which has a constant parameter α which is directly connected with the relative fluctuations within the thermodynamic limit. If this parameter is zero, one gets, as expected, the special case of the Boltzmann–Gibbs entropy. Now we will consider the case where relative fluctuations are a function of phase space volume. Because the phase space volume will grow very rapidly with increasing particle number, we assume that α is a function of $\Pi(x) = -\ln x$, i.e. fluctuations will vary slower than occupied phase space. For $x \rightarrow 0$ the value of $\Pi(x)$ diverges and on the contrary, we expect that relative fluctuations will vanish. To ensure that $\alpha(\Pi(x))$ is always finite including the limits where $\Pi(x)$ diverges, we assume that it can be represented most generally as a rational expression

$$(15) \quad \alpha(\Pi(x)) = \frac{P(\Pi(x))}{Q(\Pi(x))} = \frac{P_0 + P_1\Pi(x) + P_2\Pi^2(x) + \dots}{Q_0 + Q_1\Pi(x) + Q_2\Pi^2(x) + \dots},$$

where $P_0, P_1, P_2, \dots$ and $Q_0, Q_1, Q_2, \dots$ are constants and the numerator polynomial P has a degree of at most the degree of the denominator polynomial Q to ensure finiteness of α . For example, if Q is assumed as a linear function, one has the most general form

$$(16) \quad \alpha_{\text{linear}}(\Pi(x)) = \frac{P_0 + P_1\Pi(x)}{Q_0 + Q_1\Pi(x)} = \frac{r_0}{Q_0 + Q_1\Pi(x)} + r_1 = \frac{1}{q_0 + q_1\Pi(x)} + r_1.$$

Here, we have set $q_0 = Q_0/r_0$, $q_1 = Q_1/r_0$, $r_0 = P_0 - P_1Q_0/Q_1$, $r_1 = P_1/Q_1$. Substituting (16) into (10) yields the following ordinary differential equation

$$(17) \quad \left(\frac{1}{q_0 - q_1 \ln x} + r_1 + 1 \right) (g(x) - xg'(x)) + x^2g''(x) = 0$$

after separation of variables one obtains

$$\ln(g(x) - xg'(x)) = c_1 - \frac{\ln(q_0 - q_1 \ln x)}{q_1} + (r_1 + 1) \ln x$$

for an integration constant c_1 and after further rearranging and integrating

$$g(x) = c_2 + c_1 x \int_0^x dx' \left(x'^{(r_1-1)} (q_0 - q_1 \ln(x'))^{-\frac{1}{q_1}} \right).$$

Since x has a small value, we can make use of the mean value theorem for integration. Let $f(x) = x$ be a linear function. Then it holds

$$\begin{aligned} & \int_0^x dx' f(x') \left(x'^{(r_1-1)} (q_0 - q_1 \ln(x'))^{-\frac{1}{q_1}} \right) \\ &= f(\gamma) \int_0^x dx' \left(x'^{(r_1-1)} (q_0 - q_1 \ln(x'))^{-\frac{1}{q_1}} \right) = \int_0^x dx' \left(x'^{r_1} (q_0 - q_1 \ln(x'))^{-\frac{1}{q_1}} \right), \end{aligned}$$

where γ is a real number satisfying $0 < \gamma < x$. If we let approach x close to zero, we can assume that γ is proportional to x . This implies

$$\int_0^x dx' \left(x'^{r_1} (q_0 - q_1 \ln(x'))^{-\frac{1}{q_1}} \right) \propto x \int_0^x dx' \left(x'^{(r_1-1)} (q_0 - q_1 \ln(x'))^{-\frac{1}{q_1}} \right)$$

and we arrive at

$$g(x) = C_2 x + C_1 \int_0^x dx' \left(x'^{r_1} (q_0 - q_1 \ln(x'))^{-\frac{1}{q_1}} \right).$$

Performing the substitution $y = (q_0 - q_1 \ln(x'))^{\frac{r_1+1}{q_1}}$ and absorbing every constant in constants of integration C_1, C_2 yields

$$\begin{aligned} (18) \quad g(x) &= C_2 x + C_1 \int_{(q_0 - q_1 \ln(x'))^{\frac{r_1+1}{q_1}}}^{\infty} dy \left(e^{-y} y^{-\frac{1}{q_1}} \right) \\ &= C_2 x + C_1 \Gamma \left(1 - \frac{1}{q_1}, (q_0 - q_1 \ln(x'))^{\frac{r_1+1}{q_1}} \right). \end{aligned}$$

Here, $\Gamma(a, x) = \int_x^{\infty} dy (e^{-y} y^{a-1})$ is the upper incomplete gamma function. We have now derived the entropy associated with Hanel–Thurner exponents c and d to classify entropies. In the work by Hanel and Thurner one has additionally the parameter A which reads as $A = q_0(r_1 + 1)/q_1$ and is set for simplicity to one. This would imply the connection of physical parameters by the identity $q_0 = q_1/(r_1 + 1)$ yielding the simplified fluctuation relation

$$(19) \quad \alpha_{\text{linear}}(x) = \frac{1}{q_1} \frac{r_1 + 1}{1 - (r_1 + 1) \ln x} + r_1.$$

The Hanel–Thurner exponents are then given by $d = -1/q_1$, $c = r_1 + 1$, meaning that c is the asymptotic value of relative fluctuations when particle number would reach infinity and d is the magnitude of fluctuations that arise when particle number is finite. In particular, for the Boltzmann–Gibbs entropy (4) we have $c = 1$, $d = 1$ implying that there are no residual relative fluctuations for a system of infinite degrees of freedom, but finite-size effects will cause the system to have fluctuations. This is also a reason why the traditional Boltzmann Statistics is used even for physical systems showing significant correlations.

Tsallis entropy falls into the class $c = r_1 + 1$, $d = 0$ and thus applies for systems that have constant fluctuations regardless of size in thermodynamic limit. For example, gravitational systems are remaining always correlated, because the gravitational force is unchanged when adding other particles in contrast to, e.g., the electromagnetic force which will get screened when adding more charged particles. Another special case of (18) can be obtained when systems are regarded which have vanishing relative fluctuations in thermodynamic limit, but may exhibit finite-size correlation effects, meaning that $c = 1$, $d = -1/q_1$. Example systems are systems which have macroscopic quantum effects. Inserting these special values in (18) and fitting integration constants to axiom K3 one obtains the entropy

$$(20) \quad g_{\text{SE}} = -\Gamma\left(1 - \frac{1}{q_1}\right)x + \Gamma\left(1 - \frac{1}{q_1}, -\ln x\right).$$

The entropy (20) is also called the stretched-exponential entropy, since its equilibrium distributions will be stretched exponentials.

3. A new form of two-parameter entropy. We will now assume a stronger dependence on relative fluctuations on phase space volume. While it is expected that at extremely high phase space volumes fluctuations should fade away, the phase space volume could be greatly reduced, e.g., for constrained physical systems, even for many particles. This section assumes that α is a function of $x = 1/W(N)$ instead of a logarithmic phase space volume dependence. A simple Taylor expansion in x would lead to

$$(21) \quad \alpha = \alpha_0 + \alpha_1 x + \frac{1}{2}\alpha_2 x^2 + \dots,$$

where $\alpha_0, \alpha_1, \alpha_2, \dots$ are expansion coefficients and expression (21) will always be convergent. Again, the coefficient α_0 will capture residual fluctuations in thermodynamic limit and α_1 is the proportionality factor for finite-size fluctuation effects but with stronger phase space sensitivity than in previous considerations. Keeping terms in linear order in (21), equation (10) takes the form

$$(22) \quad (\alpha_1 x + \alpha_0 + 1)(g(x) - xg'(x)) + x^2 g''(x) = 0$$

which can be immediately integrated to

$$g(x) - xg'(x) = Cx^{\alpha_0+1}e^{\alpha_1x}.$$

Here, C is a constant of integration. Solving this ordinary differential equation leads to

$$(23) \quad g(x) = C_1x + C_2x \int_0^x dx' x'^{(\alpha_0-1)} e^{\alpha_1x'} = C_1x + C_2x(-\alpha_1)^{-\alpha_0} \gamma(\alpha_0, -\alpha_1x).$$

In (23), $\gamma(\alpha, x) = \int_0^x dy (e^{-y} y^{\alpha-1})$ is the lower incomplete gamma function and C_1, C_2 are constants of integration. The new entropy (23) has some similarities with (18), but one of their main differences is that the second argument in incomplete gamma functions scales linear in (23) instead of logarithmic in (18). Using the conditions $g(0) = 0, g(1) = 0$ one obtains the entropic form

$$(24) \quad g_{\alpha_0, \alpha_1}(x) = cx(\gamma(\alpha_0, -\alpha_1x) - \gamma(\alpha_0, -\alpha_1))$$

with a proportionality constant c . It is easy to show that $\alpha_1 \rightarrow 0$ would give Tsallis entropy. Another notable special case is obtained when setting $\alpha_0 = 0$, which is valid when the physical system does not exhibit residual fluctuations in thermodynamic limit, but will heavily fluctuate when phase space volume is small. In this case (24) simplifies to

$$(25) \quad g_{\alpha_1}(x) = cx(Ei(-\alpha_1x) - Ei(-\alpha_1)),$$

where $Ei(x) = \int_{-\infty}^{\infty} dt \frac{e^{-t}}{t}$ is the exponential integral. For sufficiently small x it will asymptotically be proportional to the logarithm, yielding the Boltzmann–Gibbs entropy. Thus, the entropic form (25) is well suited to describe systems that normally have additive entropy, but will become non-additive in entropy if they are heavily constrained. This applies for certain biological systems in which a plenty of physical observables are constrained according to evolutionary pressures that could have shaped them. Other physical systems that can be described with (24) and (25) can be quantum systems that are strongly entangled, rendering these as also being extremely constrained.

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