

RULED HYPERSURFACES IN NONFLAT COMPLEX PLANES WITH 3-KILLING STRUCTURE VECTOR FIELDS

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Abstract

Let M be a ruled hypersurface in a nonflat complex plane. In this paper, we prove that if the structure vector field of M is 3-Killing and the mean curvature of M is invariant along the structure vector field, then M is strongly 2-Hopf and the shape operator of M is completely determined.

Key words: ruled hypersurface, nonflat complex plane, 3-Killing vector field

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1. Introduction. On a Riemannian manifold (M, g) , a vector field V is said to be Killing if

$$(1.1) \quad \mathcal{L}_V g = 0,$$

where $\mathcal{L}$ denotes the Lie differentiation associated to the metric g . Or, equivalently, V is said to be Killing if the local one-parameter group generated by V consists of local isometries. V is said to be 2-Killing ([1]) if

$$(1.2) \quad \mathcal{L}_V \mathcal{L}_V g = 0.$$

Obviously, a Killing vector field is necessarily 2-Killing, but the converse is not necessarily true (for example of non-Killing 2-Killing vector fields see [1]). Very

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recently, the authors in [2] defined the notion of 3-Killing vector fields and studied it on contact Riemannian manifolds. V is said to be 3-Killing ([2]) if

$$(1.3) \quad \mathcal{L}_V \mathcal{L}_V \mathcal{L}_V g = 0.$$

Some non-Killing but 3-Killing Reeb vector fields on a contact Riemannian manifold were provided in [2].

Let $\mathbb{C}M^n(c)$ be a nonflat complex space form which is complex analytically isometric to a complex projective space $\mathbb{C}P^n(c)$ if $c > 0$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ if $c < 0$, respectively, where c denotes the constant holomorphic sectional curvature. Let M be a real hypersurface immersed in $\mathbb{C}M^n(c)$. On M , there is an almost contact metric structure (ϕ, ξ, η, g) induced from the ambient space $\mathbb{C}M^n(c)$. M is said to be Hopf if the structure vector field ξ is principal at every point of M . Hopf hypersurfaces are an important type of research objects in geometry of real hypersurfaces (see [3, 4]). A real hypersurface is said to be of type (A) if it is locally congruent to a type (A) hypersurface (for details see [3]) and they are one of most important real hypersurfaces. The structure vector field ξ of a real hypersurface in $\mathbb{C}M^n(c)$ is Killing if and only if $\phi A = A\phi$, where A is the shape operator of the hypersurface. In fact, this equality is a nice characterization of type (A) hypersurfaces.

Theorem 1.1 ([5, 6]). *The structure vector field of a real hypersurface M in a nonflat complex space form $\mathbb{C}M^n(c)$ is Killing if and only if M is of type (A).*

2-Killing vector fields on real hypersurfaces have been studied as the following.

Theorem 1.2 ([7]). *The structure vector field of a real hypersurface M in a nonflat complex space form $\mathbb{C}M^n(c)$ is 2-Killing if and only if M is of type (A).*

Unlike the above two theorems, the situation is different when the structure vector field is assumed to be 3-Killing.

Theorem 1.3 ([8]). *The structure vector field of a Hopf hypersurface M in a nonflat complex space form $\mathbb{C}M^n(c)$ is 3-Killing if and only if M is of type (A) or satisfies $\alpha^2 + c = 0$, where $\alpha = \eta(A\xi)$ is the Hopf principal curvature.*

The above three theorems related to Killing, 2-Killing or 3-Killing structure vector fields occur all on Hopf hypersurfaces and this motivated us to consider the following problem: Is there a non-Hopf hypersurface in a nonflat complex space form with 3-Killing structure vector fields? The aim of the present paper is to solve this problem on a ruled hypersurface when the ambient space is a nonflat complex plane and we get an affirmative answer.

2. Real hypersurface. Let M be a real hypersurface immersed in a complex space form $\mathbb{C}M^n(c)$ and N be a unit normal vector field of M . We denote by $\bar{\nabla}$ the Levi-Civita connection of the metric $\bar{g}$ of $\mathbb{C}M^n(c)$ and by J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of the metric g , respectively. Then the Gauss and Weingarten formulas are given respectively as the following:

$$(2.1) \quad \bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \bar{\nabla}_X N = -AX$$

for any vector fields X, Y , where A denotes the shape operator of M in $\mathbb{C}M^n(c)$. For any vector field X , we put

$$(2.2) \quad JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$(2.3) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0,$$

$$(2.4) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any vector fields X, Y . Moreover, applying the parallelism of the complex structure (i.e., $\bar{\nabla}J = 0$) of $\mathbb{C}M^n(c)$ and using (2.1), (2.2) we have

$$(2.5) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi,$$

$$(2.6) \quad \nabla_X \xi = \phi AX$$

for any vector fields X, Y . Let R be the Riemannian curvature tensor of M . Because $\mathbb{C}M^n(c)$ is of constant holomorphic sectional curvature c , the Gauss and Codazzi equations of M in $\mathbb{C}M^n(c)$ are given, respectively, as the following:

$$(2.7) \quad R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY,$$

$$(2.8) \quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}$$

for any vector fields X, Y .

We refer the reader to [3, 4] for more basic knowledge on differential geometry of real hypersurfaces in nonflat complex space forms.

3. Ruled hypersurfaces with 3-Killing structure vector field. A real hypersurface in a nonflat complex space form $\mathbb{C}M^n(c)$ is said to be ruled if

$$(3.1) \quad g(AX, Y) = 0$$

for any vector fields $X, Y \in \mathcal{D}$, where $\mathcal{D}$ denotes all vector fields orthogonal to ξ (see [3, p. 548]). Or equivalently, a real hypersurface in $\mathbb{C}M^n(c)$ is said to be ruled if the holomorphic distribution $\ker \eta$ is integrable and each of its leaves (i.e., maximal integral manifolds) is a totally geodesic complex hypersurface. A ruled hypersurface is never Hopf and in this paper we omit the singular points in a ruled hypersurface where $A\xi$ is collinear with ξ (see [9–11]). According to this, for a ruled hypersurface M in a nonflat complex plane, there exists an orthonormal frame $\{\xi, U, \phi U\}$ on M such that

$$(3.2) \quad A\xi = \alpha\xi + \beta U, \quad AU = \beta\xi, \quad A\phi U = 0,$$

where $\alpha = \eta(A\xi)$ and $\beta \neq 0$ everywhere (for example see [10, Proposition 2]). According to (3.2), the Levi-Civita connections of the hypersurface M are given as follows:

$$(3.3) \quad \begin{aligned} \nabla_U \xi &= 0, & \nabla_{\phi U} \xi &= 0, & \nabla_\xi \xi &= \beta \phi U, \\ \nabla_U U &= \kappa_1 \phi U, & \nabla_{\phi U} U &= \kappa_2 \phi U, & \nabla_\xi U &= \kappa_3 \phi U, \\ \nabla_U \phi U &= -\kappa_1 U, & \nabla_{\phi U} \phi U &= -\kappa_2 U, & \nabla_\xi \phi U &= -\kappa_3 U - \beta \xi, \end{aligned}$$

where κ_i , $i = \{1, 2, 3\}$, are three smooth functions on M defined by

$$\kappa_1 = g(\nabla_U U, \phi U), \quad \kappa_2 = g(\nabla_{\phi U} U, \phi U), \quad \kappa_3 = g(\nabla_\xi U, \phi U).$$

In fact, in view of (3.2) and (3.3), setting $X = U$ and $Y = \xi$ in the Codazzi equation (2.8) we obtain

$$(3.4) \quad U(\alpha) = \xi(\beta), \quad U(\beta) = 0, \quad \kappa_1 = \beta - \frac{c}{4\beta}.$$

Similarly, in view of (3.2) and (3.3), setting $X = \phi U$ and $Y = \xi$ in the Codazzi equation (2.8) we obtain

$$(3.5) \quad \phi U(\alpha) = \beta(\alpha + \kappa_3), \quad \phi U(\beta) = \beta^2 + \frac{c}{4}, \quad \kappa_2 = 0.$$

Moreover, according to (3.2) and the third equality in (3.5), the Lie bracket is given by

$$(3.6) \quad [\xi, U] = \kappa_3 \phi U, \quad [U, \phi U] = \left(\frac{c}{4\beta} - \beta \right) U, \quad \phi U, \xi] = \kappa_3 U + \beta \xi.$$

On a ruled hypersurface, according to (3.1) (i.e. the equivalent definition of a ruled hypersurface) we obtain $g(A\phi X, \phi Y) = 0$ for any vector fields X, Y . In view of (3.1), from this we get

$$(3.7) \quad A\phi = -\beta g_{\phi U} \otimes \xi, \quad \phi A = \beta \eta \otimes \phi U,$$

where the one-form $g_{\phi U}$ is defined by $g_{\phi U}(X) = g(\phi U, X)$ for any vector field X . With the aid of (2.6), we get

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = g((\phi A - A\phi)X, Y).$$

Inserting (3.7) into the above equality we obtain

$$(3.8) \quad \mathcal{L}_\xi g = \beta \eta \otimes g_{\phi U} + \beta g_{\phi U} \otimes \eta.$$

Taking the Lie derivative of (3.8) along the structure vector field ξ we get

$$\begin{aligned} \mathcal{L}_\xi \mathcal{L}_\xi g &= \xi(\beta) \eta \otimes g_{\phi U} + \beta(\mathcal{L}_\xi \eta) \otimes g_{\phi U} + \beta \eta \otimes (\mathcal{L}_\xi g_{\phi U}) \\ &\quad + \xi(\beta) g_{\phi U} \otimes \eta + \beta(\mathcal{L}_\xi g_{\phi U}) \otimes \eta + \beta g_{\phi U} \otimes (\mathcal{L}_\xi \eta). \end{aligned}$$

We compute the terms shown in the above equality. In fact, in view of (2.6) and (3.3), we have

$$(3.9) \quad (\mathcal{L}_\xi \eta)(X) = g(\nabla_\xi \xi, X) + g(\xi, \nabla_X \xi) = \beta g_{\phi U}(X).$$

Similarly, with the aid of (3.2) and (3.3) we have

$$(3.10) \quad (\mathcal{L}_\xi g_{\phi U})(X) = g(\nabla_\xi \phi U, X) + g(\phi U, \nabla_X \xi) = -\kappa_3 g_U(X),$$

where the one-form g_U is defined by $g_U(X) = g(U, X)$ for any vector field X . Inserting (3.9) and (3.10) into the above two equalities we have

$$(3.11) \quad \begin{aligned} \mathcal{L}_\xi \mathcal{L}_\xi g &= \xi(\beta) \eta \otimes g_{\phi U} + 2\beta^2 g_{\phi U} \otimes g_{\phi U} - \beta \kappa_3 \eta \otimes g_U \\ &\quad + \xi(\beta) g_{\phi U} \otimes \eta - \beta \kappa_3 g_U \otimes \eta. \end{aligned}$$

From this we see that the structure vector field ξ of a ruled hypersurface of dimension three is never 2-Killing because the second term in the right hand side of (3.11) is never vanishing. Again taking the Lie derivative of (3.11) along the structure vector field ξ we get

$$\begin{aligned} \mathcal{L}_\xi \mathcal{L}_\xi \mathcal{L}_\xi g &= \xi(\xi(\beta)) \eta \otimes g_{\phi U} + \xi(\beta)(\mathcal{L}_\xi \eta) \otimes g_{\phi U} + \xi(\beta) \eta \otimes (\mathcal{L}_\xi g_{\phi U}) \\ &\quad + 4\beta \xi(\beta) g_{\phi U} \otimes g_{\phi U} + 2\beta^2 (\mathcal{L}_\xi g_{\phi U}) \otimes g_{\phi U} + 2\beta^2 g_{\phi U} \otimes (\mathcal{L}_\xi g_{\phi U}) \\ &\quad - \xi(\beta \kappa_3) \eta \otimes g_U - \beta \kappa_3 (\mathcal{L}_\xi \eta) \otimes g_U - \beta \kappa_3 \eta \otimes (\mathcal{L}_\xi g_U) \\ &\quad + \xi(\xi(\beta)) g_{\phi U} \otimes \eta + \xi(\beta)(\mathcal{L}_\xi g_{\phi U}) \otimes \eta + \xi(\beta) g_{\phi U} \otimes (\mathcal{L}_\xi \eta) \\ &\quad - \xi(\beta \kappa_3) g_U \otimes \eta - \beta \kappa_3 (\mathcal{L}_\xi g_U) \otimes \eta - \beta \kappa_3 g_U \otimes (\mathcal{L}_\xi \eta). \end{aligned}$$

With the aid of (3.2) and (3.3), we have

$$(3.12) \quad (\mathcal{L}_\xi g_U)(X) = g(\nabla_\xi U, X) + g(U, \nabla_X \xi) = \kappa_3 g_{\phi U}(X).$$

Finally, inserting (3.9), (3.10) and (3.12) into the above equality we obtain

$$(3.13) \quad \begin{aligned} \mathcal{L}_\xi \mathcal{L}_\xi \mathcal{L}_\xi g &= (\xi(\xi(\beta)) - \beta \kappa_3^2) \eta \otimes g_{\phi U} + 6\beta \xi(\beta) g_{\phi U} \otimes g_{\phi U} - 3\beta^2 \kappa_3 g_U \otimes g_{\phi U} \\ &\quad - (\xi(\beta) \kappa_3 + \xi(\beta \kappa_3)) \eta \otimes g_U - 3\beta^2 \kappa_3 g_{\phi U} \otimes g_U \\ &\quad + (\xi(\xi(\beta)) - \beta \kappa_3^2) g_{\phi U} \otimes \eta - (\xi(\beta) \kappa_3 + \xi(\beta \kappa_3)) g_U \otimes \eta. \end{aligned}$$

Now we assume that the structure vector field ξ is 3-Killing. Then the right-hand side of (3.13) is vanishing and the action of it on $(\phi U, U)$ and $(\phi U, \phi U)$ gives

$$(3.14) \quad \kappa_3 = 0$$

and

$$(3.15) \quad \xi(\beta) = 0,$$

respectively. Furthermore, according to (3.2), the mean curvature is $H = \alpha/3$. If we assume that the mean curvature is invariant along the structure vector field ξ , we have $\xi(\alpha) = 0$. This, together with (3.15) and the first two equalities in (3.4), implies that both α and β are invariant along the structure vector field ξ and U .

A real hypersurface in a nonflat complex space form is said to be strongly 2-Hopf (see [12, 13]) if

- The smallest invariant distribution containing ξ under the shape operator (say $\mathcal{H} = \text{span}\{\xi, A\xi, A^2\xi, \dots\}$) is of constant rank two.
- $\mathcal{H}$ is integrable.
- The spectrum of the shape operator is invariant along $\mathcal{H}$.

In recent times, strongly (weakly) 2-Hopf hypersurfaces have been important research objects in geometry of non-Hopf real hypersurfaces in a nonflat complex space form (see [14–16]). In this situation, we observe that $\mathcal{H} = \text{span}\{\xi, U\}$ on M is of rank two and integrable. According to the above analysis we say again that both α and β are invariant along $\mathcal{H}$ and hence the hypersurface is strongly 2-Hopf. Moreover, α and β are determined by the first equality in (3.5), (3.14) and the second equality in (3.5) with $\kappa_3 = 0$, respectively. Specifically, they are given in Theorem 3.1.

Conversely, on any strongly 2-Hopf ruled hypersurface in a nonflat complex plane, from the first equality in (3.6) we have $\kappa_3 = 0$ because $\mathcal{H} = \text{span}\{\xi, U\}$ is integrable. This, together with the invariance of β along ξ , implies that ξ is 3-Killing because of (3.13). Moreover, as α is invariant along ξ , so is the mean curvature because of (3.2).

The main result of this paper is given as follows.

Theorem 3.1. *Let M be a ruled hypersurface in a nonflat complex plane $\mathbb{C}M^2(c)$. The structure vector field of M is 3-Killing and the mean curvature is invariant along the structure vector field of M if and only if M is strongly 2-Hopf. Moreover, the shape operator is given by*

$$\begin{aligned}
 & \bullet \\
 & A = \begin{pmatrix} d_1 \sec(\frac{\sqrt{c}}{2}s + d_2) & \frac{\sqrt{c}}{2} \tan(\frac{\sqrt{c}}{2}s + d_2) & 0 \\ \frac{\sqrt{c}}{2} \tan(\frac{\sqrt{c}}{2}s + d_2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ if } c > 0; \\
 & \bullet \\
 & A = \begin{pmatrix} d_1 e^{\frac{\sqrt{-c}}{2}s + d_2} & \frac{\sqrt{-c}}{2} \tanh(\frac{\sqrt{-c}}{2}s + d_2) & 0 \\ \frac{\sqrt{-c}}{2} \tanh(\frac{\sqrt{-c}}{2}s + d_2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ if } \beta = \frac{\sqrt{-c}}{2}, c < 0;
 \end{aligned}$$

$$\bullet$$

$$A = \begin{pmatrix} d_1 \operatorname{csch}(\frac{\sqrt{-c}}{2}s + d_2) & -\frac{\sqrt{-c}}{2} \coth(\frac{\sqrt{-c}}{2}s + d_2) & 0 \\ -\frac{\sqrt{-c}}{2} \coth(\frac{\sqrt{-c}}{2}s + d_2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ if } \beta > \frac{\sqrt{-c}}{2}, c < 0;$$

$$\bullet$$

$$A = \begin{pmatrix} d_1 \operatorname{sech}(\frac{\sqrt{-c}}{2}s + d_2) & -\frac{\sqrt{-c}}{2} \tanh(\frac{\sqrt{-c}}{2}s + d_2) & 0 \\ -\frac{\sqrt{-c}}{2} \tanh(\frac{\sqrt{-c}}{2}s + d_2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ if } \beta < \frac{\sqrt{-c}}{2}, c < 0,$$

where d_1, d_2 are constants and $d/ds = \phi U$.

The solution of ODEs of the first equality in (3.5) together with (3.14) and the second equality in (3.5) can be seen in literature (for example see [14, 17]). In fact, the strongly 2-Hopf ruled hypersurface was characterized in [14, Theorem 1.4] by using the assumption that the hypersurface is η -parallel and the mean curvature is invariant along the structure vector field ξ .

According to IVEY and RYAN [13, Theorem 5], if $\alpha(s)$ and $\beta(s)$ comprise a smooth solution of the ODEs:

$$(3.16) \quad \frac{d\alpha}{ds} = \alpha\beta, \quad \frac{d\beta}{ds} = \beta^2 + \frac{c}{4},$$

defined for s in an open interval $I \subset \mathbb{R}$, and such that $\beta(s)$ is non-vanishing, then there exists a real hypersurface in $\mathbb{C}P^2(c)$ or $\mathbb{C}H^2(c)$ such that its shape operator A is given by

$$A = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to a local orthonormal basis $\{\xi, U, \phi U\}$, where d/ds stands for the derivative with respect to ϕU . Therefore, one can construct a type of ruled hypersurfaces in $\mathbb{C}P^2(c)$ or $\mathbb{C}H^2(c)$ such that its structure vector field is non-Killing, non-2-Killing but 3-Killing.

REFERENCES

- [1] OPREA T. (2008) 2-Killing vector fields on Riemannian manifolds, *Balkan J. Geom. Appl.*, **13**(1), 87–92.
- [2] WANG Y., T. LIU, M. HOU (2026) Contact metric 3-manifolds with 3-Killing Reeb vector fields, submitted.
- [3] CECIL T. E., P. J. RYAN (2015) *Geometry of Hypersurfaces*, Springer Monographs in Mathematics, New York, Springer.

- [4] NIEBERGALL R., P. J. RYAN (1997) Real hypersurfaces in complex space forms, in Tight and Taut Submanifolds, Math. Sci. Res. Inst. Publ., **32**, Cambridge Univ. Press, Cambridge, 233–305.
- [5] MONTIEL S., A. ROMERO (1986) On some real hypersurfaces of a complex hyperbolic space, Geom. Dedicata, **20**(2), 245–261.
- [6] OKUMURA M. (1975) On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc., **212**, 355–364.
- [7] GHOSH A. (2015) Certain types of real hypersurfaces in complex space forms, J. Geom., **109**(1), 10, pp. 9.
- [8] WANG Y., M. HOU, T. LIU (2026) Hopf hypersurfaces in nonflat complex space forms with m -Killing structure vector fields, submitted.
- [9] KIMURA M., S. MAEDA, H. TANABE (2019) Integral curves of the characteristic vector field of minimal ruled real hypersurfaces in non-flat complex space forms, Hokkaido Math. J., **48**(3), 589–609.
- [10] MAEDA S., T. ADACHI (2006) Integral curves of characteristic vector fields of real hypersurfaces in nonflat complex space forms, Geom. Dedicata, **123**(1), 65–72.
- [11] MAEDA S., H. NAITOH (2011) Real hypersurfaces with ϕ -invariant shape operator in a complex projective space, Glasgow Math. J., **53**(2), 347–358.
- [12] DÍAZ-RAMOS J. C., M. DOMÍNGUEZ-VÁZQUEZ, C. VIDAL-CASTIÑEIRA (2018) Strongly 2-Hopf hypersurfaces in complex projective and hyperbolic planes, Ann. Mat. Pura Appl., **197**(2), 469–486.
- [13] IVEY T. A., P. J. RYAN (2016) Hypersurfaces in $\mathbb{CP}^2$ and $\mathbb{CH}^2$ with two distinct principal curvatures, Glasgow Math. J., **58**(1), 137–152.
- [14] CUI Q., Z. HU (2023) On 2-Hopf hypersurfaces in nonflat complex planes, Mediterr. J. Math., **20**(6), 324, pp. 23.
- [15] CUI Q., Z. HU (2025) On non-Hopf Ricci-pseudosymmetric hypersurfaces in $\mathbb{CP}^2$ and $\mathbb{CH}^2$, Math. Nachr., **298**(2), 527–547.
- [16] WANG Y., Y. ZHOU (2025) Levi-umbilical weakly 2-Hopf hypersurface in nonflat complex planes, J. Math. Anal. Appl., **546**(1), 129255, pp. 9.
- [17] TANABE H., S. MAEDA (2012) Sectional curvatures of ruled real hypersurfaces in a nonflat complex space form, Proc. Workshop on Differential Geometry of Submanifolds and Its Related Topics, Saga, 113–118.

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