

EXISTENCE RESULTS FOR A CLASS OF SINGULAR
PROBLEMS INVOLVING $p(x)$ -LAPLACIAN OPERATOR

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Received on December 11, 2025

Presented by P. Popivanov, Member of BAS, on March 31, 2026

Abstract

The purpose of this work is to study a class of $p(x)$ -Laplacian type equations involving variable singularities. By using the Fountain theorem together with Mountain pass theorem, we show the existence and multiplicity of solutions for the strong ($q \in (1, \infty)$) variable singularities.

Key words: variational methods, $p(x)$ -Laplacian type equation, Mountain pass theorem, Fountain theorem, strong variable singularity

2020 Mathematics Subject Classification: 35J75, 35A21, 35A15, 35J65

1. Introduction. The aim of this paper is to study the following singular $p(x)$ -Laplacian type equation with Hardy type term

$$(P) \quad \begin{cases} -\operatorname{div}(a(x, \nabla u)) = \frac{|u|^{q(x)-2}u}{w^{2q(x)}} + \lambda f(x, u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth bounded domain; $\lambda > 0$ is a real parameter, the weight w is, for instance, $w(x) := |x|$ or $w(x) := \operatorname{dist}(x, \partial\Omega)$, $q \in C(\overline{\Omega})$ such that $1 < q(x) < \infty$ for any $x \in \overline{\Omega}$, $\operatorname{div}(a(x, \nabla u))$ is a $p(x)$ -Laplacian type operator, $p \in C(\overline{\Omega})$ such that $1 < p^- := \inf_{x \in \overline{\Omega}} p(x)$ for any $x \in \overline{\Omega}$ and $a(x, \xi): \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$

is the continuous derivative with respect to ξ of the mapping $A: \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}$,

<https://doi.org/10.7546/CRABS.2026.04.02>

$A = A(x, \xi)$, i.e., $a(x, \xi) = \nabla_{\xi} A(x, \xi)$. Moreover, $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function.

Throughout this paper, we take into account the following set of hypotheses:

(A1) There exists a constant $c_0 > 0$ such that

$$|a(x, \xi)| \leq c_0(1 + |\xi|^{p(x)-1}) \text{ for all } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^N.$$

(A2) The following inequalities hold

$$|\xi|^{p(x)} \leq a(x, \xi) \cdot \xi \leq p(x)A(x, \xi) \text{ for all } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^N.$$

(A3) $A(x, 0) = 0$ for all $x \in \overline{\Omega}$.

(A4) $A(x, -\xi) = A(x, \xi)$ for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$.

(A5) There exists a constant $k_0 > 0$ such that for all $x \in \overline{\Omega}$ and $u, v \in \mathbb{R}^N$

$$A\left(x, \frac{u+v}{2}\right) \leq \frac{1}{2}A(x, u) + \frac{1}{2}A(x, v) - k_0|u-v|^{p(x)}.$$

The study of differential equations and variational problems with singular nonlinearities has been very attractive due to the application areas of these studies. Such problems arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials. Moreover, these equations are also encountered in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contexts [1–4].

There are many results on the existence (multiple positive) of solutions for p -Laplacian and p -biharmonic problems with singular nonlinearities for Dirichlet problems; we refer to [5–7]. However, since studies on the existence of positive solutions for $p(x)$ -Laplacian problems with singular nonlinearities are rare, we could find these results in only a few papers [8–13]. For instance, in [8], using a variational method and Krasnoselskii's genus theory with the classical Hardy's inequality, the authors investigated the existence and multiplicity of the solutions for a singular elliptic problem. Moreover, they studied closedness of set of eigenfunctions such that $p(x) \equiv p$. In [12], the authors proved the existence of two weak solutions and at least three weak solutions, using Bonanno's theorem with the classical Hardy's inequality for a singular elliptic problem. In [9–11, 13], these studies dealt with singular $p(x)$ -Kirchhoff-type equations and have studied the case where the singularity takes values in the range $(0, 1)$, i.e., the weak singularity, except for papers [9, 11]. In [9], the author obtained the existence and uniqueness of a positive solution for the case of variable singularity with values

in the interval $(1, \infty)$, i.e., strong singularity, using Ekeland's variational principle and a constrained minimization approach. However, in [11], the authors are concerned with the existence and infinite solutions to a singular $p(x)$ -Kirchhoff-type problem with a Hardy-type (singular) term w , which is a weight function in the form $w(x) := |x|$ or $w(x) := \text{dist}(x, \partial\Omega)$, by using Mountain Pass and the Krasnoselskii's theory. Inspired by the above studies, the problem **(P)** was studied.

The paper is organized as follows. In Section 2, we state some elementary properties concerning the variable exponent Lebesgue and Sobolev spaces which we will use later. Then, we present our main results. In Section 3, we provide the existence and the multiplicity of the solution.

2. Preliminaries and main results. In order to deal with problem **(P)**, we recall some definitions and basic properties of the generalized Lebesgue Sobolev spaces $(L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega))$; we refer to [14–16]. Suppose that Ω is a bounded open domain of \mathbb{R}^N and set

$$C_+(\bar{\Omega}) = \{p: p \in C(\bar{\Omega}), p(x) > 1 \text{ for all } x \in \bar{\Omega}\}.$$

For $p(x) \in C_+(\bar{\Omega})$ we define $1 < p^- := \inf_{x \in \bar{\Omega}} p(x) \leq p(x) \leq p^+ := \sup_{x \in \bar{\Omega}} p(x) < \infty$.

The variable exponent Lebesgue space is defined as

$$L^{p(x)}(\Omega) = \left\{ u \mid u: \Omega \rightarrow \mathbb{R} \text{ is a measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$

On this space we define a norm, the so-called Luxemburg norm, by the formula

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{(p(x), \Omega)} = \inf \left\{ \vartheta > 0: \int_{\Omega} \left| \frac{u(x)}{\vartheta} \right|^{p(x)} dx \leq 1 \right\}.$$

Define also the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega): |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{1,p(x)} := \inf \left\{ \zeta > 0: \int_{\Omega} \left(\left| \frac{\nabla u(x)}{\zeta} \right|^{p(x)} + \left| \frac{u(x)}{\zeta} \right|^{p(x)} \right) dx \leq 1 \right\}$$

or

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)} \text{ for all } u \in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. We can define an equivalent norm

$$\|u\| = |\nabla u|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces if and only if $1 < p^- \leq p^+ < \infty$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$.

Proposition 2.1 ([15]). *The Hölder inequality type holds, namely*

$$\left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)}$$

for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

The convex functional $\phi(u) : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\phi(u) := \int_{\Omega} |u(x)|^{p(x)} \, dx$$

is called modular on $L^{p(x)}(\Omega)$.

Proposition 2.2 ([14, 15]). *If $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \dots$, and $p^+ < \infty$, then the following properties hold:*

- (i) $\|u\|_{p(x)} > 1 (= 1, > 1) \iff \phi(u) > 1 (= 1, > 1)$,
- (ii) $\min(\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}) \leq \phi(u) \leq \max(\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+})$,
- (iii) $\|u_n - u\|_{p(x)} \rightarrow 0 \ (\rightarrow \infty) \iff \phi(u_n - u) \rightarrow 0 \ (\rightarrow \infty)$.

Proposition 2.3 ([17]).

(i) *if $m(x) \in C_+(\overline{\Omega})$ and $m(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega)$ is compact and continuous, where*

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } N > p(x), \\ \infty, & \text{if } N \leq p(x); \end{cases}$$

(ii) *Poincaré inequality; i.e., there is a positive constant $C > 0$ such that*

$$\|u\|_{p(x)} \leq C \|u\| \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

Lemma 2.4 (Mountain-Pass Geometry [18]). *Let X be a Banach space and let $J \in C^1(X, \mathbb{R})$ satisfy Palais-Smale condition. Assume that $J(0) = 0$.*

- (i) *There exist two positive real numbers τ and r such that $J(u) \geq r > 0$ with $\|u\| = \tau$.*
- (ii) *There exists $u_1 \in X$ such that $\|u_1\| > \tau$ and $J(u_1) < 0$.*

Put

$$G = \{\Gamma \in C([0, 1], X) : \Gamma(0) = 0, \Gamma(1) = u_1\}.$$

Set $\beta = \inf\{\max J_\lambda(\Gamma([0, 1])) : \Gamma \in G\}$. Then $\beta \geq r$ and β is a critical value of J .

Since X is a reflexive and separable Banach space, then there exist (see [19, Sect. 17]) $e_j \subset X$ and $e_j^* \subset X^*$ such that

$$X = \overline{\text{span}\{e_j \mid j = 1, 2, \dots\}}, \quad X^* = \overline{\text{span}\{e_j^* \mid j = 1, 2, \dots\}}$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between X and X^* . For convenience, we write

$$X_j = \text{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^k X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

Lemma 2.5 ([14]). *If $\alpha(x) \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$ for any $x \in \overline{\Omega}$, denote*

$$\beta_k = \sup\{\|u\|_{L^{\alpha(x)}(\Omega)} : \|u\| = 1, u \in Z_k\}.$$

Then $\lim_{k \rightarrow \infty} \beta_k = 0$.

Lemma 2.6 (Fountain Theorem [18]). *Assume that X is a Banach space, $J \in C^1(X, \mathbb{R})$ is an even functional satisfying the (PS). Moreover, if there exists $\rho_k > \gamma_k > 0$ for each $k = 1, 2, \dots$, such that*

- (i) $\inf_{u \in Z_k, \|u\| = \gamma_k} J(u) \rightarrow \infty$ as $k \rightarrow \infty$,
- (ii) $\max_{u \in Y_k, \|u\| = \rho_k} J_\lambda(u) \leq 0$,
- (iii) J satisfies (PS) condition for every $c > 0$,

then J has a sequence of critical values tending to $+\infty$.

Theorem 2.7. *Assume that (A1)–(A5) and f is a Carathéodory function satisfying the following conditions:*

(f₁) *There exists a positive constant c_1 such that*

$$|f(x, t)| \leq c_1(1 + |t|^{\alpha(x)-1}) \text{ for all } (x, t) \in \Omega \times \mathbb{R},$$

where $\alpha(x) \in C_+(\overline{\Omega})$ satisfies $p^+ < \alpha^- < \alpha(x) < p^*(x)$ for each $x \in \Omega$.

(f₂) $f(x, t) = o(|t|^{p^+-1})$ as $t \rightarrow 0$ for $x \in \Omega$ and $p^+ < \alpha^-$.

(AR) *There exists $t^* > 0$ and $\theta \in (p^+, q^-)$ such that*

$$0 < \theta F(x, t) \leq f(x, t)t, \quad |t| \geq t^* \text{ for all } (x, t) \in \Omega \times \mathbb{R},$$

where $F(x, t) = \int_0^t f(x, s) ds$.

Then for any $\lambda \in (0, \infty)$ problem (P) has a nontrivial weak solution in $W_0^{1,p(x)}(\Omega)$.

Theorem 2.8. Assume that (A1)–(A5) and f satisfies the following condition

$$(f_3) \quad f(x, -t) = -f(x, t) \text{ for all } (x, t) \in \Omega \times \mathbb{R}.$$

Then for any $\lambda \in (0, \infty)$ problem (P) has infinite many pairs of solutions in $W_0^{1,p(x)}(\Omega)$.

3. Proofs of the main results. Let X denote the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$. The singular functional $J_\lambda: X \rightarrow \mathbb{R}$ for $u \in X$ associated with problem (P), defined by

$$J_\lambda(u) = \Lambda(u) - \int_{\Omega} \frac{1}{q(x)} \frac{|u|^{q(x)}}{w(x)^{2q(x)}} dx - \int_{\Omega} F(x, u) dx, \quad \Lambda(u) = \int_{\Omega} A(x, \nabla u) dx.$$

We say that $u \in X$ is a weak solution of problem (P) if and only if

$$\int_{\Omega} a(x, \nabla u) \nabla v = \int_{\Omega} \frac{|u|^{q(x)-2} u}{w(x)^{2q(x)}} v dx + \lambda \int_{\Omega} f(x, u) v dx$$

for any $v \in X$.

Lemma 3.1 ([10]). *If $p, q \in C(\bar{\Omega})$ satisfy $1 < p^- \leq p(x) \leq p^+ < q^- \leq q^+ < \min\{p^*(x), N/2\}$ for all $x \in \bar{\Omega}$, then there exist two positive constants $H, C > 0$ such that the $(p(\cdot), q(\cdot))$ -Hardy inequality*

$$\int_{\Omega} \frac{|u|^{q(x)}}{w(x)^{2q(x)}} dx \leq \frac{1}{H} \int_{\Omega} |\nabla u(x)|^{p(x)} + C^{q^+} \|u(x)\|^{q^+}$$

holds for all $u \in X$.

Remark 3.2. In this paper the weight function w is assumed to be either $w(x) := |x|$ or $w(x) := \text{dist}(x, \partial\Omega)$. These two cases correspond to different Hardy-type inequalities. If $w(x) := |x|$, we assume that $\Omega \subset B_R(0)$ for some $H := R > 0$; in this case the corresponding Hardy-type inequality involves a constant depending on R . On the other hand, when $w(x) := \text{dist}(x, \partial\Omega)$, the Hardy inequality holds with a positive constant $H := R_\Omega$ which depends on the geometry of the domain. In general, this constant is not explicitly known.

Lemma 3.3 ([20]).

(i) A verifies the growth condition

$$|A(x, \xi)| \leq c_0(|\xi| + |\xi|^{p(x)})$$

for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^N$.

(ii) A is $p(x)$ -homogeneous, i.e.,

$$A(x, t_1 \xi) \leq A(x, \xi) t_1^{p(x)}$$

for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^N$ and $t_1 \geq 1$.

Lemma 3.4 ([20]).

- (i) *The functional Λ is well defined on X .*
- (ii) *The functional Λ is of class $C^1(X, \mathbb{R})$ and*

$$\langle \Lambda'(u), v \rangle = \int_{\Omega} \Omega a(x, \nabla u) \cdot \nabla v \, dx \quad \text{for all } u, v \in X.$$

- (iii) *The functional Λ is weakly lower semi-continuous on X .*
- (iv) *For all $u, v \in X$*

$$\Lambda\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(v) - k_0 \|u-v\|^{p^-}.$$

- (v) *For all $u, v \in X$*

$$\Lambda(u) - \Lambda(v) \geq \langle \Lambda'(v), u-v \rangle.$$

- (vi) *J_λ is weakly lower semi-continuous on X .*

From Lemmas 3.3, 3.4 and **(f₁)** it is easy to see that $J_\lambda \in C^1(X, \mathbb{R})$ and the derivative of J_λ is the mapping $J'_\lambda: X \rightarrow \mathbb{R}$. Then, we obtain

$$\langle J'_\lambda(u), v \rangle = \int_{\Omega} a(x, \nabla u) \nabla v \, dx - \int_{\Omega} \frac{|u|^{q(x)-2} u}{w(x)^{2q(x)}} v \, dx - \lambda \int_{\Omega} f(x, u) v \, dx,$$

for any $u, v \in X$. So, weak solutions of problem **(P)** correspond to critical points of the functional J_λ .

Lemma 3.5. *If **(f₁)**, **(A2)** and **(AR)** hold, then for any $\lambda \in (0, +\infty)$, the functional J_λ satisfies Palais–Smale (PS) condition.*

Proof. Let assume that there exists a sequence $\{u_n\} \subset X$ such that

$$(3.1) \quad |J(u_n)| \leq c \quad \text{and} \quad J'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Firstly, we claim that $\{u_n\}$ is bounded in X . If it is not true, by passing to a subsequence, if necessary, we may assume that $\|u_n\| \rightarrow \infty$. Without loss of generality, we assume that $\|u_n\| > 1$, for n large enough. By using **(AR)** and (3.1), we obtain

$$\begin{aligned} c + \|u_n\| &\geq J_\lambda(u_n) - \frac{1}{\theta} \langle J'_\lambda(u_n), u_n \rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \int_{\Omega} |\nabla u_n|^{p(x)} \, dx - \frac{1}{q^-} \int_{\Omega} \frac{|u_n|^{p(x)}}{w(x)^{2q(x)}} \, dx + \frac{1}{\theta} \int_{\Omega} \frac{|u_n|^{q(x)}}{w(x)^{q(x)}} \, dx \\ &\quad + \lambda \int_{\Omega} \left(\frac{1}{\theta} f(x, u_n) u_n - F(x, u_n)\right) \, dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \|u_n\|^{p^-} + \left(\frac{1}{\theta} - \frac{1}{q^-}\right) \int_{\Omega} \frac{|u_n|^{q(x)}}{w(x)^{2q(x)}} \, dx. \end{aligned}$$

By using **(A2)**, Proposition 2.2 together with the condition $\theta \in (p^+, q^-)$, we get

$$c + \|u_n\| \geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p^-}.$$

If the last inequality is divided by $\|u_n\|^{p^-}$ and passes to the limit as $n \rightarrow \infty$, we obtain a contradiction to the conditions $p^- > 1$ and $\theta \in (p^+, q^-)$. In this situation, $\{u_n\}$ is bounded in X . Therefore, there is a subsequence $\{u_n\} \subset X$ and $u \in X$ such that $\{u_n\}$ converges weakly to u in X .

Since $\alpha(x) < p^*(x)$ and $q(x) < p^*(x)$, we conclude from Proposition 2.3 that X is compactly embedded in the spaces $L^{\alpha(x)}(\Omega)$ and $L^{q(x)}(\Omega)$, that is,

$$(3.2) \quad \begin{aligned} u_n &\rightarrow u \text{ in } L^{\alpha(x)}(\Omega), & u_n &\rightarrow u \text{ a.e. on } \Omega, \\ u_n &\rightarrow u \text{ in } L^{q(x)}(\Omega), & u_n &\rightarrow u \text{ a.e. on } \Omega. \end{aligned}$$

To complete the proof, it remains to show that $u_n \rightarrow u$ in X . Taking into account relation (3.1), we can write that $\langle J'_\lambda(u_n), u_n - u \rangle \rightarrow 0$. That is

$$\begin{aligned} \langle J'_\lambda(u_n), u_n - u \rangle &= \int_{\Omega} a(x, \nabla u_n)(\nabla u_n - \nabla u) \, dx \\ &\quad - \int_{\Omega} \frac{|u_n|^{q(x)-2} u_n}{w(x)^{2q(x)}} (u_n - u) \, dx - \int_{\Omega} f(x, u_n)(u_n - u) \, dx \rightarrow 0. \end{aligned}$$

From Proposition 2.1, it follows that

$$\left| \int_{\Omega} \frac{|u_n|^{q(x)-2} u_n}{w(x)^{2q(x)}} (u_n - u) \, dx \right| \leq c_2 \left| \frac{|u_n|^{q(x)-1}}{w(x)^{2q(x)}} \right|_{L^{q'(x)}} \|u_n - u\|_{L^{q(x)}(\Omega)}.$$

Similarly, using **(f1)** and Proposition 2.1, we have

$$\begin{aligned} \left| \int_{\Omega} f(x, u_n)(u_n - u) \, dx \right| &\leq \int_{\Omega} \left| \left(c_1 + c_1 |u_n|^{\alpha(x)-1} \right) \right| |u_n - u| \, dx \\ &\leq c_3 \left(1 + \left| |u_n|^{\alpha(x)-1} \right|_{L^{\alpha'(x)}} \right) \|u_n - u\|_{L^{\alpha(x)}(\Omega)}, \end{aligned}$$

where c_2 and $c_3 > 0$ are constants. From (3.2), $\{u_n\}$ converges strongly to u in $L^{\alpha(x)}(\Omega)$ and $L^{q(x)}(\Omega)$, that is, $\|u_n - u\|_{L^{\alpha(x)}(\Omega)} \rightarrow 0$ and $\|u_n - u\|_{L^{q(x)}(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. So, we obtain

$$(3.3) \quad \int_{\Omega} f(x, u_n)(u_n - u) \, dx \rightarrow 0$$

and

$$(3.4) \quad \int_{\Omega} \frac{|u_n|^{q(x)-2} u_n}{w(x)^{2q(x)}} (u_n - u) \, dx \rightarrow 0.$$

By combining (3.3) and (3.4) and using the fact that $\langle J'_\lambda(u_n), u_n - u \rangle \rightarrow 0$, we conclude that

$$\int_{\Omega} a(x, \nabla u_n)(\nabla u_n - \nabla u) dx \rightarrow 0.$$

That is, $\lim_{n \rightarrow \infty} \langle \Lambda'(u_n), u_n - u \rangle = 0$. By using Lemma 3.4(v), we have

$$0 = \lim_{n \rightarrow \infty} \langle \Lambda'(u_n), u - u_n \rangle \leq \lim_{n \rightarrow \infty} (\Lambda(u) - \Lambda(u_n)) = \Lambda(u) - \lim_{n \rightarrow \infty} \Lambda(u_n)$$

or

$$(3.5) \quad \lim_{n \rightarrow \infty} \Lambda(u_n) \leq \Lambda(u).$$

So, from Lemma 3.4(iii) and (3.5), we obtain $\lim_{n \rightarrow \infty} \Lambda(u_n) = \Lambda(u)$.

Now, we assume by contradiction that $\{u_n\}$ does not converge strongly to u in X . Then, there exist $\zeta > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\|u_{n_k} - u\| \geq \zeta$. Moreover, by Lemma 3.4(iv), we get

$$(3.6) \quad \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(u_{n_k}) - \Lambda\left(\frac{u_{n_k} + u}{2}\right) \geq k_0\|u_{n_k} - u\|^{p^-} \geq k_0\zeta^{p^-}.$$

Passing to the limit as k tends to infinity in the inequality 3.6, we have

$$\limsup_{n \rightarrow \infty} \Lambda\left(\frac{u_{n_k} + u}{2}\right) \leq \Lambda(u) - k_0\zeta^{p^-}.$$

Moreover, we have that $\left\{\frac{u_{n_k} + u}{2}\right\}$ converges weakly to u in X . From Lemma 3.4 (iii), we obtain

$$\Lambda(u) \leq \liminf_{n \rightarrow \infty} \Lambda\left(\frac{u_{n_k} + u}{2}\right),$$

which is a contradiction. Hence, it follows that $\{u_n\}$ converges strongly to u in X . The proof is complete. \square

Lemma 3.6. *Suppose (\mathbf{f}_1) , (\mathbf{f}_2) , $(\mathbf{A1})$ – $(\mathbf{A5})$ and (\mathbf{AR}) are satisfied. Then the following statements hold:*

(i) *There exist two positive real numbers τ and r such that $J_\lambda(u) \geq r$, $u \in X$ with $\|u\| = \tau$.*

(ii) *There exists $u \in X$ such that $\|u\| > \tau$, $J_\lambda(u) < 0$.*

Proof. (i) Let $\|u\| < 1$. Since X is continuously embedded in $L^{p^+}(\Omega)$ and $L^{\alpha(x)}(\Omega)$, and according to Proposition 2.3, there exist positive constants c_4 and c_5 such that

$$(3.7) \quad |u|_{L^{\alpha(x)}(\Omega)} \leq c_4\|u\| \quad \text{and} \quad |u|_{L^{p^+}(\Omega)} \leq c_5\|u\|, \quad u \in X.$$

From (\mathbf{f}_1) and (\mathbf{f}_2) , we obtain

$$(3.8) \quad F(x, t) \leq \varepsilon |t|^{p^+} + c_\varepsilon |t|^{\alpha(x)}, \quad (x, t) \in \Omega \times \mathbb{R}.$$

Therefore, by Proposition 2.2, Lemma 3.1, (3.7) and (3.8), we can deduce that

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+} \|u\|^{p^+} - \frac{1}{Hq^-} \int_\Omega |\nabla u|^{p(x)} - \frac{C^{q^+}}{q^-} \|u\|^{q^+} - \lambda \int_\Omega (\varepsilon |u|^{p^+} + c_\varepsilon |u|^{\alpha(x)}) dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{Hq^-} \right) \|u\|^{p^+} - \frac{C^{q^+}}{q^-} \|u\|^{q^+} - \lambda c_5^{p^+} \|u\|^{p^+} - \lambda c_\varepsilon c_4^{\alpha^-} \|u\|^{\alpha^-} \end{aligned}$$

for $\|u\|$ small enough. Let $\varepsilon > 0$ be small enough such that

$$0 < \lambda c_5^{p^+} \leq \frac{1}{2} \left(\frac{1}{p^+} - \frac{1}{Hq^-} \right)$$

and if we can choose $p^+ < Hq^-$, we write

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{2} \left(\frac{1}{p^+} - \frac{1}{Hq^-} \right) \|u\|^{p^+} - \left(\frac{C^{q^+}}{Hq^-} + \lambda c_\varepsilon c_4^{\alpha^-} \right) \|u\|^{\alpha^-} \\ &\geq \|u\|^{p^+} \left(\frac{1}{2} \left(\frac{1}{p^+} - \frac{1}{Hq^-} \right) - c_6 \|u\|^{\alpha^- - p^+} \right). \end{aligned}$$

Since $p^+ < \alpha^-$, then we can choose $\|u\| = \tau$ small enough such that the functional $h: [0, 1] \rightarrow \mathbb{R}$ defined by

$$h(t) = \frac{1}{2} \left(\frac{1}{p^+} - \frac{1}{Hq^-} \right) - c_6 \tau^{\alpha^- - p^+}$$

is positive on neighbourhood of the origin. It follows that there exist two positive real numbers τ and r such that $J_\lambda(u) \geq r$, $u \in X$ with $\|u\| = \tau \in (0, 1)$.

(ii) From (\mathbf{AR}) we obtain

$$(3.9) \quad F(x, t) \geq c_7 |t|^\theta, \quad |t| \geq t^* \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$ and $\psi \neq 0$. Moreover, using together Lemma 3.3 and inequalities (3.9), we obtain

$$\begin{aligned} J_\lambda(t\psi) &= \int_\Omega A(x, \nabla t\psi) dx - \frac{1}{q^+} \int_\Omega \frac{|t\psi|^{q(x)}}{w(x)^{2q(x)}} dx - \int_\Omega F(x, t\psi) dx \\ &\leq |t|^{p^+} \int_\Omega A(x, \nabla \psi) dx - \frac{|t|^{q^-}}{q^+} \int_\Omega |\psi|^{p(x)} dx - \lambda c_8 |t|^\theta \int_\Omega |\psi|^\theta dx. \end{aligned}$$

As $\theta \in (p^+, q^-)$, we infer that $J_\lambda(t\psi) \rightarrow -\infty$ as $t \rightarrow +\infty$ for any $\lambda \in (0, \infty)$. Then, there exists $u_1 = t\psi$ such that $\|u_1\| > \tau$ and $J_\lambda(u_1) < 0$. The proof is complete. \square

Proof of Theorem 2.7. From Lemma 3.5, Lemma 3.6, and the fact that $J_\lambda(0) = 0$ from (A3), J_λ satisfies all conditions of Lemma 2.4. Thus, J_λ has at least one nontrivial weak solution point, i.e., problem (P) has a nontrivial weak solution. \square

We will use *Fountain theorem* to prove Theorem 2.8.

Proof of Theorem 2.8. It is enough to show that J_λ has an unbounded sequence of critical points. Because J_λ is an even functional from the assumptions (f₃) and (A4), and J_λ satisfies (PS) condition from Lemma 3.5, we only need to show whether it satisfies conditions (i) and (ii) in Lemma 2.6.

(i) For any $u \in Z_k$ such that $\|u\| = \gamma_k > 1$, from (f₁) and Lemma 3.1, we have

$$\begin{aligned} J_\lambda(u) &\geq \left(\frac{1}{p^+} - \frac{1}{Hq^-}\right) \|u\|^{p^-} - \frac{C^{q^+}}{q^-} \|u\|^{q^+} - \lambda c_1 \int_{\Omega} \left(1 + |u|^{\alpha(x)-1}\right) dx, \\ &\geq \left(\frac{1}{p^+} - \frac{1}{Hq^-}\right) \|u\|^{p^-} - \frac{C^{q^+}}{q^-} \|u\|^{q^+} - \lambda c_1 \max \left\{ |u|_{L^{\alpha(x)}(\Omega)}^{\alpha^+}, |u|_{L^{\alpha(x)}(\Omega)}^{\alpha^-} \right\} - c_9. \end{aligned}$$

If $\max\{\|u\|^{q^+}, \|u\|^{\alpha^+}\} = \|u\|^{\alpha^+}$ and $p^+ < Hq^-$, then by Proposition 2.2, we obtain

$$\begin{aligned} J_\lambda(u) &\geq \begin{cases} \left(\frac{1}{p^+} - \frac{1}{Hq^-}\right) \|u\|^{p^-} - \lambda c_{10} - c_9, & \text{if } |u|_{L^{\alpha(x)}(\Omega)} \leq 1, \\ \left(\frac{1}{p^+} - \frac{1}{Hq^-}\right) \|u\|^{p^-} - \frac{C^{q^+}}{q^-} \|u\|^{q^+} \\ - \lambda c_{10} \beta_k^{\alpha^+} \|u\|^{\alpha^+} - c_9, & \text{if } |u|_{L^{\alpha(x)}(\Omega)} > 1 \end{cases} \\ &\geq \left(\frac{1}{p^+} - \frac{1}{Hq^-}\right) \|u\|^{p^-} - \lambda c_{11} \beta_k^{\alpha^+} \|u\|^{\alpha^+} - c_9, \end{aligned}$$

where $c_9, c_{10}, c_{11} > 0$ are constants. For $\|u\| = \gamma_k = \left(\frac{2p^+ Hq^-}{Hq^- - p^+} \lambda c_{11} \beta_k^{\alpha^+}\right)^{\frac{1}{p^- - \alpha^+}}$, we get

$$J_\lambda(u) \geq \frac{1}{2} \left(\frac{1}{p^+} - \frac{1}{Hq^-}\right) \gamma_k^{p^-} - c_9.$$

Since $\beta_k \rightarrow 0$ and $p^- < \alpha^+$, we obtain $\gamma_k \rightarrow \infty$ as $k \rightarrow \infty$. Consequently, we deduce that $J_\lambda(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$ for $u \in Z_k$. The statement of (i) is satisfied.

(ii) From (AR), there exist constants $c_{12}, c_{13} > 0$ such that

$$F(x, t) \geq c_{12}|t|^\theta - c_{13}.$$

Let $u \in Y_k$ with $\|u\| = \rho_k > 1$. Then we write

$$J_\lambda(u) \leq \frac{1}{p^-} \|u\|^{p^+} - \frac{1}{Hq^-} \|u\|^{p^+} - \frac{C^{q^+}}{Hq^-} \|u\|^{q^+} - \lambda c_{12} \int_{\Omega} |u|^\theta dx + c_{13}.$$

Since the space Y_k has finite dimension, $|\cdot|_{L^\theta(\Omega)}$ and $\|\cdot\|$ are equivalent norms in Y_k . Then, we obtain

$$J_\lambda(u) \leq \frac{1}{p^-} \|u\|^{p^+} - \frac{1}{Hq^-} \|u\|^{p^+} - \frac{Cq^+}{Hq^-} \|u\|^{q^+} - \lambda c_{12}^\theta \|u\|^\theta + c_{13}.$$

Consequently, we have

$$J_\lambda(u) \rightarrow -\infty \quad \text{as } n \rightarrow +\infty \quad \text{for any } u \in Y_k$$

because $\theta \in (p^+, q^-)$. We show that for each $k = 1, 2, \dots$ there exists $\rho_k > \gamma_k > 0$ such that

$$\max_{u \in Y_k, \|u\| = \rho_k} J_\lambda(u) \leq 0.$$

The assertion (ii) is satisfied and the proof of Theorem 2.8 is complete. \square

Acknowledgments. We thank all reviewers for their constructive suggestions.

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