Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 79, No 5, 2026

MATHEMATICS
Numerical methods

QUASI-MONTE CARLO INTEGRATION IN THE MIXED KOROBOV–SOBOLEV TYPE SPACE $H_{Mix,s,\alpha,\gamma,\mathrm{w}}$

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Received on March 14, 2026
Presented by S. Margenov, Corresponding Member of BAS, on March 31, 2026

Abstract

In this article, the authors construct a special reproducing kernel Hilbert space of mixed Korobov–Sobolev type. To address the main problems, an appropriate hybrid function system is employed. The arithmetic associated with this function system is used to define the notion of the mean square worst-case error of the integration in the considered space. An explicit formula for this error expressed in terms of the functions of the employed system is obtained.

Key words: mixed Korobov–Sobolev type space, hybrid function system, mean square worst-case error, exact formula

2020 Mathematics Subject Classification: 65C05, 65C20

1. Introduction. Following Aronszajn [1], we recall the notion of a reproducing kernel for a Hilbert space. Let F be a class of functions defined on E that forms a Hilbert space. The function $K(x, y)$ of $x, y \in E$ is called a *reproducing kernel* for the space F if the following properties hold:

1) For every fixed $y \in E$ the kernel $K(x, y)$ as a function of x belongs to F;

2) (*reproducing property*) For each function $f \in F$ and an arbitrary $y \in E$ the equality $f(y) = \langle f(x),\, K(x, y)\rangle_x$ holds. Here, the subscript x indicates that the inner product is given with respect to the variable x.

Let $s \geq 1$ be a fixed integer and denote the dimension throughout the article. The technique of numerical integration in reproducing kernel Hilbert spaces is as

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`https://doi.org/10.7546/CRABS.2026.05.01`

follows: let $H_s(K)$ be a Hilbert space generated by the reproducing kernel K with an inner product $\langle\cdot\rangle_{H_s(K)}$. The integral

$$I_s(f) = \int_{[0,1]^s} f(\mathbf{x})d\mathbf{x}, \ f \in H_s(K)$$

is approximated by a Quasi-Monte Carlo algorithm

$$Q_s(f; P_N) = \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n),$$

where $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ is a deterministic sample point net in $[0,1)^s$. The *worst-case error* of the integration in the space $H_s(K)$ by using the net P_N of nodes is defined as

$$e(H_s(K); P_N) = \sup_{f \in H_s(K),\ ||f||_{H_s(K)} \leq 1} |I_s(f) - Q_s(f; P_N)|.$$

The main advantage of the Quasi-Monte Carlo algorithm in reproducing kernel Hilbert spaces is that there exists an explicit formula for the worst-case error of the integration. Thus, following SLOAN and WOŹNIAKOWSKI [2], the worst-case error of the integration in the Hilbert space $H_s(K)$ by using an arbitrary net P_N of nodes satisfies the equality

$$(1) \qquad e(H_s(K); P_N)$$
$$= \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{2}{N}\sum_{n=0}^{N-1}\int_{[0,1]^s} K(\mathbf{x}_n, \mathbf{y})d\mathbf{y} + \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} K(\mathbf{x}_n, \mathbf{x}_m).$$

The Hilbert spaces can be divided into two main types – Korobov and Sobolev spaces. In general, the Korobov spaces are sets of functions on which conditions on the speed of incline to zero of their Fourier's coefficients are imposed. The Sobolev spaces are sets of functions on which conditions on smoothness of their mixed partial derivatives of a given order are imposed.

There are two main approaches to constructing Hilbert spaces: without using weights and with using weights for successive arguments. In these cases, the so-called unweighted and weighted Hilbert spaces are obtained. In the unweighted case, the weights are usually assumed to be equal to one.

The idea of using weights was introduced by Sloan and Woźniakowski [2] and describes the dependence of the functions on their successive arguments. We assume that for a function $f(x_1, \ldots, x_s)$ the variable x_1 is the most essential, x_2 is less essential and so on, x_s is the smallest essential. This is technically realized through using weights $\gamma_1, \ldots, \gamma_s$ such that $1 \geq \gamma_1 \geq \ldots \geq \gamma_s > 0$, which are associated with the arguments of the function.

In this article, we construct a Hilbert space of mixed Korobov–Sobolev type, which combines the concepts of Korobov and Sobolev spaces into one focus. For our investigations, we use a special kind of hybrid function system. The arithmetic associated with this hybrid system allows us to introduce the notion of the mean square worst-case error of integration and a formula in explicit form for this error will be obtained.

2. A construction of the hybrid function system. For each vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{Z}^s$ the function $e_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ is defined as $e_{\mathbf{k}}(\mathbf{x}) = e^{2\pi \mathbf{i}(k_1x_1+\ldots+k_sx_s)}$, $\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$. The set $\mathcal{T}^s = \{e_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{Z}^s,\ \mathbf{x} \in [0,1)^s\}$ is called the trigonometric system.

For a function $f \in L_1([0,1)^s)$ and a vector $\mathbf{k} \in \mathbb{Z}^s$ the $\mathbf{k}$-th Fourier's coefficient of f is defined as $\widehat{f}_{\mathcal{T}^s}(\mathbf{k}) = \displaystyle\int_{[0,1]^s} f(\mathbf{x}).\overline{e}_{\mathbf{k}}(\mathbf{x})d\mathbf{x}$. Here, for an arbitrary complex number z the symbol $\overline{z}$ denotes the conjugated number of z.

For each reals $x, y \in [0,1)$ we define $x \oplus_{10}^{[0,1)} y = x + y \pmod 1$ and for vectors $\mathbf{x} = (x_1, \dots, x_s)$, $\mathbf{y} = (y_1, \dots, y_s) \in [0,1)^s$ we put $\mathbf{x} \oplus_{10}^{[0,1)^s} \mathbf{y} = (x_1 \oplus_{10}^{[0,1)} y_1, \dots, x_s \oplus_{10}^{[0,1)} y_s)$. The subscript 10 indicates that the operation $\oplus_{10}^{[0,1)^s}$ is performed in the decimal system.

The so-called Cantor systems are natural generalizations of the ordinary b-adic number system. Thus, let $B = \{b_0, b_1, \dots : b_i \geq 2 \text{ for } i \geq 0\}$ be a sequence of integers. To the set B corresponds the sequence $\{B_0, B_1, \dots\}$ of the so-called *generalized powers*, which are defined as: $B_0 = 1$ and for $i \geq 0$ $B_{i+1} = B_i.b_i$. For the Cantor system with the above components we use the notion of the B-adic number system. An arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ have B-adic representations of the form $k = \displaystyle\sum_{i=0}^{\nu} k_iB_i$ and $x = \displaystyle\sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$, where for $i \geq 0$ $k_i, x_i \in \{0, 1, \dots, b_i - 1\}$ and $k_\nu \neq 0$. The representation of k is unique and under the additional assumptions that for infinitely many indices i we have that $x_i \neq b_i - 1$, the representation of x is also unique.

For $1 \leq j \leq s$ let $B_j = \{b_{j,0}, b_{j,1}, \dots : b_{j,i} \geq 2 \text{ for } i \geq 0\}$ be an arbitrary sequence of integers. To each sequence B_j corresponds the set of generalized powers $\{B_{j,0}, B_{j,1}, \dots\}$. Let us denote $\mathcal{B}_s = (B_1, \dots, B_s)$ and to use the terminology $\mathcal{B}_s$-adic Cantor system.

In 2021 PETROVA [3] by using an arbitrary $\mathcal{B}_s$-adic Cantor system, proposed the construction of the functions of the system $\Gamma_{\mathcal{B}_s}$. Thus, for an arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ with the B-adic representations $k = \displaystyle\sum_{i=0}^{\nu} k_iB_i$ and $x = \displaystyle\sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$, the k-th function ${}_B\gamma_k : [0,1) \to \mathbb{C}$ is defined as ${}_B\gamma_k(x) = e^{2\pi \mathbf{i}\left(\frac{k_0}{B_1}+\ldots+\frac{k_\nu}{B_{\nu+1}}\right)(x_0+x_1B_1+\ldots+x_\nu B_\nu)}$.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denote the set of non-negative integers. For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-th function ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ is defined as

$$ {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{B_j}\gamma_{k_j}(x_j), \ \mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s. $$

The set $\Gamma_{\mathcal{B}_s} = \{{}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s, \ \mathbf{x} \in [0,1)^s\}$ is called the $\mathcal{B}_s$-adic function system.

For arbitrary reals $x, y \in [0,1)$ with the B-adic representations $x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$ and $y = \sum_{i=0}^{\infty} \frac{y_i}{B_{i+1}}$ the operation $\oplus_B^{[0,1)}$ is defined as $x \oplus_B^{[0,1)} y = \sum_{i=0}^{\infty} \frac{z_i}{B_{i+1}}$, where the sequence $\{z_0, z_1, \dots\}$ is determined by the following recursive rules:

$$ x_0 + y_0 = t_0.b_0 + z_0, \ t_0 \in \{0,1\}, \ z_0 \in \{0,1,\dots,b_0-1\}, $$

$$ x_1 + y_1 + t_0 = t_1.b_1 + z_1, \ t_1 \in \{0,1\}, \ z_1 \in \{0,1,\dots,b_1-1\} \text{ and so on.} $$

For arbitrary vectors $\mathbf{x} = (x_1, \dots, x_s), \ \mathbf{y} = (y_1, \dots, y_s) \in [0,1)^s$ the operation $\oplus_{\mathcal{B}_s}^{[0,1)^s}$ is defined as $\mathbf{x} \oplus_{\mathcal{B}_s}^{[0,1)^s} \mathbf{y} = (x_1 \oplus_{B_1}^{[0,1)} y_1, \dots, x_s \oplus_{B_s}^{[0,1)} y_s)$.

To develop the concept of the hybrid function system, we first present some preliminary notations. Let $s \geq 1$ be a fixed dimension and s_1, s_2 be arbitrary integers such that $0 \leq s_1, s_2 \leq s$ and $s_1 + s_2 = s$. The numbers s_1 and s_2 we will call *subdimensions*. Let us denote $\mathcal{H}^s = \mathbb{Z}^{s_1} \times \mathbb{N}_0^{s_2}$ and for an arbitrary vector $\mathbf{k} \in \mathcal{H}^s$ let us use the denotation $\mathbf{k} = (\mathbf{k}^{(1)}, \mathbf{k}^{(2)})$, where $\mathbf{k}^{(1)} \in \mathbb{Z}^{s_1}$ and $\mathbf{k}^{(2)} \in \mathbb{N}_0^{s_2}$ are subdimensional coordinates of $\mathbf{k}$. For an arbitrary vector $\mathbf{x} \in [0,1)^s$ let us use the denotation $\mathbf{x} = (\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$, where for $\tau = 1, 2$ $\mathbf{x}^{(1)} \in [0,1)^{s_\tau}$.

Let $\mathcal{B}_{s_2}$ be an arbitrary Cantor system. We define the hybrid function system $\mathcal{F}$ as a tensor product of the subdimensional systems $\mathcal{T}^{s_1}$ and $\Gamma_{\mathcal{B}_{s_2}}$, i.e. $\mathcal{F} = \mathcal{T}^{s_1} \otimes \Gamma_{\mathcal{B}_{s_2}}$. This means that for arbitrary vectors $\mathbf{k}$ and $\mathbf{x}$ as above, the function ${}_{\mathcal{F}}h_{\mathbf{k}}(\mathbf{x}) \in \mathcal{F}$ is given in the form ${}_{\mathcal{F}}h_{\mathbf{k}}(\mathbf{x}) = e_{\mathbf{k}^{(1)}}(\mathbf{x}^{(1)}).{}_{\mathcal{B}_{s_2}}\gamma_{\mathbf{k}^{(2)}}(\mathbf{x}^{(2)})$. Then the set $\mathcal{F}$ is given as $\mathcal{F} = \{{}_{\mathcal{F}}h_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathcal{H}^s, \ \mathbf{x} \in [0,1)^s\}$. GROZDANOV [4] proved that *the hybrid function system $\mathcal{F}$ is a complete orthonormal basis of the space $L_2([0,1)^s)$*.

Let us denote $\mathcal{B} = (10, \mathcal{B}_{s_2})$. For arbitrary vectors $\mathbf{x} = (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}), \ \mathbf{y} = (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}) \in [0,1)^s$ let us define the operation

$$ \mathbf{x} \oplus_{\mathcal{B}}^{[0,1)^s} \mathbf{y} = (\mathbf{x}^{(1)} \oplus_{10}^{[0,1)^{s_1}} \mathbf{y}^{(1)}, \ \mathbf{x}^{(2)} \oplus_{\mathcal{B}_{s_2}}^{[0,1)^{s_2}} \mathbf{y}^{(2)}). $$

We will note the important fact that for each function $f \in L_1([0,1)^s)$ and an arbitrary fixed vector $\sigma \in [0,1)^s$ the equality holds

$$ \int_{[0,1]^s} f(\mathbf{x} \oplus_{\mathcal{B}}^{[0,1)^s} \sigma) d\mathbf{x} = \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x}. \tag{2} $$

3. A construction of the mixed Korobov–Sobolev type space $\boldsymbol{H_{Mix,s,\alpha,\gamma,\mathbf{w}}}$. Now, we present the constructive principle of the subdimensional weighted Hilbert spaces. For this purpose, for $\tau = 1, 2$ let $\gamma^{(\tau)} = (\gamma_1^{(\tau)}, \ldots, \gamma_{s_\tau}^{(\tau)})$, where $1 \geq \gamma_1^{(\tau)} \geq \ldots \geq \gamma_{s_\tau}^{(\tau)} > 0$, be arbitrary vectors of coordinate weights and to denote $\gamma = (\gamma^{(1)}, \gamma^{(2)})$.

For arbitrary reals $\alpha > 1$, $\gamma > 0$ and an integer k let us define the coefficient

$$R(\alpha;\gamma;k) = \begin{cases} 1, & \text{if} \quad k = 0, \\ \dfrac{\gamma}{|k|^\alpha}, & \text{if} \quad k \neq 0. \end{cases}$$

For arbitrary vectors $\alpha = (\alpha_1, \ldots, \alpha_{s_1})$, where for $1 \leq j \leq s_1$ $\alpha_j > 1$ of exponential parameters, $\gamma^{(1)}$ of weights and $\mathbf{k}^{(1)} = (k_1^{(1)}, \ldots, k_{s_1}^{(1)}) \in \mathbb{Z}^{s_1}$ let us define the coefficients $R(\alpha;\gamma^{(1)};\mathbf{k}^{(1)}) = \displaystyle\prod_{j=1}^{s_1} R(\alpha_j;\gamma_j^{(1)};k_j^{(1)})$.

For two functions f and g defined over $[0,1)^{s_1}$ let us define their inner product as

$$\langle f, g \rangle_{s_1,\alpha,\gamma^{(1)}} = \sum_{\mathbf{k}^{(1)} \in \mathbb{Z}^{s_1}} R^{-1}(\alpha;\gamma^{(1)};\mathbf{k}^{(1)}).\widehat{f}_{\mathcal{T}^{s_1}}(\mathbf{k}^{(1)}).\overline{\widehat{g}}_{\mathcal{T}^{s_1}}(\mathbf{k}^{(1)}).$$

Then the Korobov space $H_{s_1,\alpha,\gamma^{(1)}}$ is given as $H_{s_1,\alpha,\gamma^{(1)}} = \{f : ||f||_{s_1,\alpha,\gamma^{(1)}} < +\infty\}$.

Let us construct the function

$$\begin{aligned} (3) \quad & K_{s_1,\alpha,\gamma^{(1)}}(\mathbf{x}^{(1)}, \mathbf{y}^{(1)}) \\ & = \sum_{\mathbf{k}^{(1)} \in \mathbb{Z}^{s_1}} R(\alpha;\gamma^{(1)};\mathbf{k}^{(1)}).e_{\mathbf{k}^{(1)}}(\mathbf{x}^{(1)}).\overline{e}_{\mathbf{k}^{(1)}}(\mathbf{y}^{(1)}), \ \mathbf{x}^{(1)}, \mathbf{y}^{(1)} \in [0,1)^{s_1}. \end{aligned}$$

Lemma 1. *The function $K_{s_1,\alpha,\gamma^{(1)}}$ is the reproducing kernel of the space $H_{s_1,\alpha,\gamma^{(1)}}$.*

Following HICKERNEL [5], see also NOVAK and WOŹNIAKOWSKI [6], we recall the concept of the so-called anchored Sobolev space $H_{Sob,s_2,\gamma^{(2)},\mathbf{w}}$. Let $\mathbf{w} = (w_1, \ldots, w_{s_2})$, where for $1 \leq j \leq s_2$ $w_j \in [0,1]$, be an arbitrary vector, which is called an *anchor*.

For two functions f and g defined over $[0,1)^{s_2}$ their inner product is defined as

$$\begin{aligned} & \langle f, g \rangle_{s_2,\gamma^{(2)},\mathbf{w}} \\ & = \sum_{u \subseteq \{1,\ldots,s_2\}} [\gamma_u^{(2)}]^{-1} \int_{[0,1]^{|u|}} \frac{\partial^{|u|}}{\partial \mathbf{x}_u^{(2)}} f(\mathbf{x}_u^{(2)}, \mathbf{w}_{-u}) \frac{\partial^{|u|}}{\partial \mathbf{x}_u^{(2)}} g(\mathbf{x}_u^{(2)}, \mathbf{w}_{-u}) d\mathbf{x}_u^{(2)}. \end{aligned}$$

Here, for an arbitrary subset u of $\{1, \ldots, s_2\}$ and vectors $\mathbf{x}^{(2)} = (x_1^{(2)}, \ldots, x_{s_2}^{(2)})$ and $\mathbf{w}$ the symbols $\mathbf{x}_u^{(2)}$ denote a vector with coordinates $x_j^{(2)}$ when $j \in u$ and

$\mathbf{w}_{-u}$ – a vector with coordinates w_j when $j \notin u$. The symbol $|u|$ denotes the cardinality of the set u. Then the space $H_{Sob,s_2,\gamma^{(2)},\mathbf{w}}$ is defined as $H_{Sob,s_2,\gamma^{(2)},\mathbf{w}} = \{||f||_{s_2,\gamma^{(2)},\mathbf{w}} < +\infty\}$.

For arbitrary reals $\gamma > 0$ and $w \in [0,1]$ the function is constructed

$$K_{1,\gamma,w}(x,y) = 1 + \frac{\gamma}{2}(|x-w| + |y-w| - |x-y|),\ x,y \in [0,1).$$

For vectors $\gamma^{(2)}$ and $\mathbf{w}$ also the function

$$K_{s_2,\gamma^{(2)},\mathbf{w}}(\mathbf{x}^{(2)},\mathbf{y}^{(2)}) = \prod_{j=1}^{s_2} K_{1,\gamma_j^{(2)},w_j}(x_j^{(2)}, y_j^{(2)}),$$

$\mathbf{x}^{(2)} = (x_1^{(2)},\ldots,x_{s_2}^{(2)})$, $\mathbf{y}^{(2)} = (y_1^{(2)},\ldots,y_{s_2}^{(2)}) \in [0,1)^{s_2}$ is constructed and it is proved that *the function $K_{s_2,\gamma^{(2)},\mathbf{w}}$ is the reproducing kernel for the space $H_{Sob,s_2,\gamma^{(2)},\mathbf{w}}$*.

In our work we will use the spaces $H_{s_1,\alpha,\gamma^{(1)}}$ and $H_{Sob,s_2,\gamma^{(2)},\mathbf{w}}$ to define the space $H_{Mix,s,\alpha,\gamma,\mathbf{w}} = H_{s_1,\alpha,\gamma^{(1)}} \otimes H_{Sob,s_2,\gamma^{(2)},\mathbf{w}}$, which is a space of mixed Korobov–Sobolev type. This means that

$$f(\mathbf{x}) \in H_{Mix,s,\alpha,\gamma,\mathbf{w}} \Leftrightarrow f(\mathbf{x}) = f_1(\mathbf{x}^{(1)}).f_2(\mathbf{x}^{(2)}),\ \ \mathbf{x} = (\mathbf{x}^{(1)},\mathbf{x}^{(2)}) \in [0,1)^s,$$

$$f_1(\mathbf{x}^{(1)}) \in H_{s_1,\alpha,\gamma^{(1)}},\ \mathbf{x}^{(1)} \in [0,1)^{s_1},\ \ f_2(\mathbf{x}^{(2)}) \in H_{Sob,s_2,\gamma^{(2)},\mathbf{w}},\ \mathbf{x}^{(2)} \in [0,1)^{s_2}.$$

For two functions $f, g \in H_{Mix,s,\alpha,\gamma,\mathbf{w}}$ to define their inner product as

$$\langle f,g\rangle_{s,\alpha,\gamma,\mathbf{w}} = \langle f_1,g_1\rangle_{s_1,\alpha,\gamma^{(1)}}.\langle f_2,g_2\rangle_{s_2,\gamma^{(2)},\mathbf{w}}.$$

The last equality shows us that $H_{Mix,s,\alpha,\gamma,\mathbf{w}} = \{f : \|f\|_{s,\alpha,\gamma,\mathbf{w}} < +\infty\}$.

For arbitrary vectors $\mathbf{x} = (\mathbf{x}^{(1)},\mathbf{x}^{(2)})$, $\mathbf{y} = (\mathbf{y}^{(1)},\mathbf{y}^{(2)}) \in [0,1)^s$ let us define the function

$$K_{s,\alpha,\gamma,\mathbf{w}}(\mathbf{x},\mathbf{y}) = K_{s_1,\alpha,\gamma^{(1)}}(\mathbf{x}^{(1)},\mathbf{y}^{(1)}).K_{s_2,\gamma^{(2)},\mathbf{w}}(\mathbf{x}^{(2)},\mathbf{y}^{(2)}).$$

Lemma 2. *The function $K_{s,\alpha,\gamma,\mathbf{w}}$ is the reproducing kernel for the space $H_{Mix,s,\alpha,\gamma,\mathbf{w}}$.*

4. The mean square worst-case error of the integration in the space $\boldsymbol{H_{Mix,s,\alpha,\gamma,\mathbf{w}}}$. Let $P_N = \{\mathbf{x}_0,\ldots,\mathbf{x}_{N-1}\}$ be an arbitrary net of points in $[0,1)^s$ and $\sigma \in [0,1)^s$ be an arbitrary and fixed vector. We will use the operation $\oplus_{\mathcal{B}}^{[0,1)^s}$ to define the net $P_N(\sigma) = \{\mathbf{x}_0 \oplus_{\mathcal{B}}^{[0,1)^s} \sigma,\ldots,\mathbf{x}_{N-1} \oplus_{\mathcal{B}}^{[0,1)^s} \sigma\}$, which we call *$\mathcal{B}$-adic digitally shifted net.* We will give the following definition:

Definition 1. Let K be an arbitrary reproducing kernel. We will define the notion of the associated $\mathcal{B}$-adic digitally shifted kernel as

$$K_{\mathcal{B}-ds}(\mathbf{x},\mathbf{y}) = \int_{[0,1]^s} K(\mathbf{x} \oplus_{\mathcal{B}}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{\mathcal{B}}^{[0,1)^s} \sigma)d\sigma,\ \mathbf{x},\ \mathbf{y} \in [0,1)^s.$$

Definition 2. Let $H_s(K)$ be an arbitrary Hilbert space, P_N an arbitrary net of points in $[0,1)^s$, $\sigma \in [0,1)^s$ be a fixed vector, and $P_N(\sigma)$ be the $\mathcal{B}$-adic digitally shifted net. Let $e(H_s(K); P_N(\sigma))$ be the worst-case error of the integration in the space $H_s(K)$ by using the net $P_N(\sigma)$. We define the quantity

$$\tilde{e}^2(H_s(K); P_N) = \int_{[0,1]^s} e^2(H_s(K); P_N(\sigma))d\sigma,$$

which we call mean square worst-case error of the integration in the space $H_s(K)$ by using the net P_N.

Theorem 1. *Let $H_s(K)$ be an arbitrary Hilbert space and P_N a net of points in $[0,1)^s$. Then the mean square worst-case error of the integration in the space $H_s(K)$ by using the net P_N satisfies the equality $\tilde{e}(H_s(K); P_N) = e(H_s(K_{\mathcal{B}-ds}); P_N)$, i.e. the mean square worst-case error of the integration in the space $H_s(K)$ is equal to the ordinary worst-case error of the integration in the space $H_s(K_{\mathcal{B}-ds})$ generated by the $\mathcal{B}$-adic digitally shifted kernel $K_{\mathcal{B}-ds}$ and by using the same net P_N.*

Proof. According to Definition 2 and equality (1) we obtain the presentation

$$\tilde{e}^2(H_s(K); P_N) = \int_{[0,1]^s} e^2(H_s(K); P_N(\sigma))d\sigma \tag{4}$$

$$= \int_{[0,1]^s} \left[\int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{2}{N}\sum_{n=0}^{N-1} \int_{[0,1]^s} K(\mathbf{x}_n \oplus_{\mathcal{B}}^{[0,1)^s} \sigma, \mathbf{y})d\mathbf{y} \right.$$

$$\left. + \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} K(\mathbf{x}_n \oplus_{\mathcal{B}}^{[0,1)^s} \sigma, \mathbf{x}_m \oplus_{\mathcal{B}}^{[0,1)^s} \sigma) \right] d\sigma.$$

To rewrite the expressions in the equality above, we will use property (2) and Definition 1. Thus, the following holds:

$$\int_{[0,1]^s} \left[\int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} \right] d\sigma \tag{5}$$

$$= \int_{[0,1]^s} \left[\int_{[0,1]^{2s}} K(\mathbf{x} \oplus_{\mathcal{B}}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{\mathcal{B}}^{[0,1)^s} \sigma)d\mathbf{x}d\mathbf{y} \right] d\sigma$$

$$= \int_{[0,1]^{2s}} \left[\int_{[0,1]^s} K(\mathbf{x} \oplus_{\mathcal{B}}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{\mathcal{B}}^{[0,1)^s} \sigma)d\sigma \right] d\mathbf{x}d\mathbf{y} = \int_{[0,1]^{2s}} K_{\mathcal{B}-ds}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y};$$

$$\int_{[0,1]^s} \left[\frac{2}{N}\sum_{n=0}^{N-1} \int_{[0,1]^s} K(\mathbf{x}_n \oplus_{\mathcal{B}}^{[0,1)^s} \sigma, \mathbf{y})d\mathbf{y} \right] d\sigma \tag{6}$$

$$= \frac{2}{N} \sum_{n=0}^{N-1} \int_{[0,1]^s} \left[\int_{[0,1]^s} K(\mathbf{x}_n \oplus_{\mathcal{B}}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{\mathcal{B}}^{[0,1)^s} \sigma) d\mathbf{y} \right] d\sigma$$

$$= \frac{2}{N} \sum_{n=0}^{N-1} \int_{[0,1]^s} \left[\int_{[0,1]^s} K(\mathbf{x}_n \oplus_{\mathcal{B}}^{[0,1)^s} \sigma, \mathbf{y} \oplus_{\mathcal{B}}^{[0,1)^s} \sigma) d\sigma \right] d\mathbf{y}$$

$$= \frac{2}{N} \sum_{n=0}^{N-1} \int_{[0,1]^s} K_{\mathcal{B}-ds}(\mathbf{x}_n, \mathbf{y}) d\mathbf{y};$$

$$(7) \qquad \int_{[0,1]^s} \left[\frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} K(\mathbf{x}_n \oplus_{\mathcal{B}}^{[0,1)^s} \sigma, \mathbf{x}_m \oplus_{\mathcal{B}}^{[0,1)^s} \sigma) \right] d\sigma$$

$$= \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \left[\int_{[0,1]^s} K(\mathbf{x}_n \oplus_{\mathcal{B}}^{[0,1)^s} \sigma, \mathbf{x}_m \oplus_{\mathcal{B}}^{[0,1)^s} \sigma) d\sigma \right]$$

$$= \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} K_{\mathcal{B}-ds}(\mathbf{x}_n, \mathbf{x}_m).$$

From equalities (1), (4), (5), (6), and (7) we finally obtain that

$$\tilde{e}^2(H_s(K); P_N) = \int_{[0,1]^{2s}} K_{\mathcal{B}-ds}(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} - \frac{2}{N} \sum_{n=0}^{N-1} \int_{[0,1]^s} K_{\mathcal{B}-ds}(\mathbf{x}_n, \mathbf{y}) d\mathbf{y}$$

$$+ \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} K_{\mathcal{B}-ds}(\mathbf{x}_n, \mathbf{x}_m) = e^2(H_s(K_{\mathcal{B}-ds}); P_N).$$

□

Let us choose $s_2 = 0$, i.e. $s_1 = s$. In this case, the hybrid operation $\oplus_{\mathcal{B}}^{[0,1)^s}$ and the notion of the $\mathcal{B}$-adic digitally shifted kernel $K_{\mathcal{B}-ds}$ will be reduced to the operation $\oplus_{10}^{[0,1)^{s_1}}$ and the notion of the 10-adic digitally shifted kernel K_{10-ds}. Definition 1 and equality (3) imply that for $\forall\ \mathbf{x},\ \mathbf{y} \in [0,1)^{s_1}$ the equality $K_{10-ds,s_1,\alpha,\gamma^{(1)}}(\mathbf{x}, \mathbf{y}) = K_{s_1,\alpha,\gamma^{(1)}}(\mathbf{x}, \mathbf{y})$ holds. From Theorem 1 the equality $\tilde{e}(H_{s_1}(K_{s_1,\alpha,\gamma^{(1)}}); P_N) = e(H_{s_1}(K_{s_1,\alpha,\gamma^{(1)}}); P_N)$ is obtained. Hence, the notion mean square worst-case error of the integration in the Korobov space $H_{s_1,\alpha,\gamma^{(1)}}$ coincides with the ordinary worst-case error of the integration in this space.

The general theory of the reproducing kernels states that if K is an arbitrary reproducing kernel, then for $\forall\ \mathbf{x},\ \mathbf{y} \in [0,1)^s$

$$K_{\mathcal{B}-ds}(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{k} \in \mathcal{H}^s} {}_{\mathcal{F}}\widehat{K}(\mathbf{k}, \mathbf{k}) . {}_{\mathcal{F}}h_{\mathbf{k}}(\mathbf{x}) . {}_{\mathcal{F}}\overline{h}_{\mathbf{k}}(\mathbf{y}),$$

where for each vector $\mathbf{k} \in \mathcal{H}^s$ the equality

$$_{\mathcal{F}}\widehat{K}(\mathbf{k}, \mathbf{k}) = \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}).{}_{\mathcal{F}}\overline{h}_{\mathbf{k}}(\mathbf{x}).{}_{\mathcal{F}}h_{\mathbf{k}}(\mathbf{y}) d\mathbf{x} d\mathbf{y}$$

holds. In the case when $K_s(\mathbf{x}, \mathbf{y}) = K_{s_1}(\mathbf{x}^{(1)}, \mathbf{y}^{(1)}).K_{s_2}(\mathbf{x}^{(2)}, \mathbf{y}^{(2)})$, the above remark is also valid for the kernels K_{s_1} and K_{s_2}, but with respect to the systems $\mathcal{T}^{s_1}$ and $\Gamma_{\mathcal{B}_{s_2}}$.

Lemma 3. *The following holds:*

(*i*) *For each vector* $\mathbf{k}^{(1)} \in \mathbb{Z}^{s_1}$ *the equality*

$$_{\mathcal{T}^{s_1}}\widehat{K}_{s_1,\alpha,\gamma^{(1)}}(\mathbf{k}^{(1)}, \mathbf{k}^{(1)}) = R(\alpha, \gamma^{(1)}; \mathbf{k}^{(1)})$$

holds;

(*ii*) *Let B be an arbitrary Cantor system and for each integer $k \geq 1$ to use the B-adic representation $k = k_g B_g + k_{g-1} B_{g-1} + \ldots + k_1 B_1 + k_0$, where $g \geq 0$, for $0 \leq i \leq g$ $k_i \in \{0, 1, \ldots, b_i - 1\}$ and $k_g \neq 0$. Then for each integer $k \geq 0$ the equalities hold*

$$_{\Gamma_B}\widehat{K}_{1,\gamma,w}(k, k) = \begin{cases} 1 + \gamma \left(w^2 - w + \dfrac{1}{3} \right), & \text{if} \quad k = 0, \\ \dfrac{\gamma}{2} \left(\dfrac{1}{\sin^2 \pi \frac{k_g}{b_g}} - \dfrac{1}{3} \right) \dfrac{1}{B_{g+1}^2}, & \text{if} \quad k \geq 1. \end{cases}$$

For each vector $\mathbf{k}^{(2)} = (k_1^{(2)}, \ldots, k_{s_2}^{(2)}) \in \mathbb{N}_0^{s_2}$ *the equality holds*

$$_{\Gamma_{\mathcal{B}_{s_2}}}\widehat{K}_{s_2,\gamma^{(2)},\mathbf{w}}(\mathbf{k}^{(2)}, \mathbf{k}^{(2)}) = \prod_{j=1}^{s_2} {}_{\Gamma_{B_j^{(2)}}}\widehat{K}_{1,\gamma_j^{(2)},w_j}(k_j^{(2)}, k_j^{(2)});$$

(*iii*) *The associated $\mathcal{B}$-adic digitally shifted kernel $K_{\mathcal{B}-ds,s,\alpha,\gamma,\mathbf{w}}$ has a Fourier's representation of the form*

$$K_{\mathcal{B}-ds,s,\alpha,\gamma,\mathbf{w}}(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{k} \in \mathcal{H}^s} {}_{\mathcal{F}}\widehat{K}_{s,\alpha,\gamma,\mathbf{w}}(\mathbf{k}, \mathbf{k}).{}_{\mathcal{F}}h_{\mathbf{k}}(\mathbf{x}).{}_{\mathcal{F}}\overline{h}_{\mathbf{k}}(\mathbf{y}), \ \forall \ \mathbf{x}, \ \mathbf{y} \in [0,1)^s,$$

where for each vector $\mathbf{k} = (\mathbf{k}^{(1)}, \mathbf{k}^{(2)}) \in \mathcal{H}^s$ the equality holds ${}_{\mathcal{F}}\widehat{K}_{s,\alpha,\gamma,\mathbf{w}}(\mathbf{k}, \mathbf{k}) = {}_{\mathcal{T}^{s_1}}\widehat{K}_{s_1,\alpha,\gamma^{(1)}}(\mathbf{k}^{(1)}, \mathbf{k}^{(1)}).{}_{\Gamma_{\mathcal{B}_{s_2}}}\widehat{K}_{s_2,\gamma^{(2)},\mathbf{w}}(\mathbf{k}^{(2)}, \mathbf{k}^{(2)})$.

Theorem 2. *Let $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net of points in $[0,1)^s$. Then the mean square worst-case error of the integration in the space $H_{Mix,s,\alpha,\gamma,\mathbf{w}}$ by using the net P_N satisfies the equality*

$$\tilde{e}^2(H_{Mix,s,\alpha,\gamma,\mathbf{w}}; P_N) = -\prod_{j=1}^{s_2} \left[1 + \gamma_j^{(2)} \left(w_j^2 - w_j + \frac{1}{3} \right) \right]$$

$$+\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{\mathbf{k}\in\mathcal{H}^s} {}_{\mathcal{F}}\widehat{K}_{s,\alpha,\gamma,\mathbf{w}}(\mathbf{k},\mathbf{k}).{}_{\mathcal{F}}h_{\mathbf{k}}(\mathbf{x}_n).{}_{\mathcal{F}}\overline{h}_{\mathbf{k}}(\mathbf{x}_m).$$

The construction of Hilbert spaces of mixed Korobov–Sobolev type can be further extended, particularly with respect to Sobolev spaces. For example, one may employ the so-called unanchored Sobolev space. To implement this idea, the hybrid system must be extended with an additional function system – such as the Vilenkin system constructed over an arbitrary Cantor system. In this regard, a more general form of arithmetic can be defined, leading to a new expression for the mean square worst-case error of the integration. Consequently, this approach yields a result more general than that presented in Theorem 2.

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